Differential Equations for High School Students

Preparation for University Studies in Engineering, Mathematics, Physics and Allied Fields.

February 23, 2002

P. Michael Henderson

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Preface

If you're a high school student reading this paper, you're probably fairly intelligent and have done well in your high school mathematics courses. Otherwise, you wouldn't have sought out a paper on differential equations.

As you progress in your education and in life, remember that doing well in school and in mathematics is not sufficient for success in life. You'll find that success in life is like a pyramid – there are lots of people in the first layer, fewer in the next layer, and so forth. Only a small number of people can make it to the top, in whatever field you choose. How do the people who make it to the top get there?

To be successful in life, you need to excel in several areas.

1. **You must be intelligent and creative**. You don't have to be the most intelligent person. In fact, my experience is that the most intelligent person often doesn't make it to the top because s/he relies too much on intelligence, and intelligence is only one factor in success. Real success requires that you be "good" in a number of areas, the rest of which are described in the following bullet items.

2. **You must have a high capacity for work and a strong drive to succeed**. At the highest levels, the difference between success and failure is usually very small. The person who pushes the hardest, who achieves a slight "edge," usually wins.

3. **You need to get along with people**. Success is not an individual accomplishment. Work is accomplished in teams, with people sharing ideas and contributing to the overall success. You need to do more than just "get along" with people. The most successful people are those who other people like to be with and whom they seek out to talk about work and share things with. When you want to accomplish something and have to convince others to follow you, it'll be easier if the people like and respect you.

4. **You must take prudent risks**. This is the place where most people are weak. Taking risks is not something most of us are taught – we're taught to be obedient and to do what we're told. But in business (and in your personal life) you only get ahead by taking risks and making things happen. The risks must be prudent. You must think through the situation and have a realistic belief that you can succeed. And once you commit to a course of action, put all of yourself into it and make it happen. A less-than-optimal course of action backed up with hard work and commitment will beat out any half-heartedly supported plan.

Success requires a balance between all of the above characteristics. If you're lacking in any one of them, it's unlikely that you will find real success – unrewarded genius is an old familiar story. So strive for a balance in life. If you're not the smartest in your class, you may have the foundation for success where it really counts – in life.

But enough advice. I hope that this paper will assist you in understanding your engineering courses in college.

Preparing a paper of this size is a significant task, requiring preparation and planning, and lots of time. The writing of this paper had to be squeezed into my days already filled with work and family responsibilities. I want to thank my wife, Norma, for her understanding and support during the time I worked on it. She never questioned my need or desire to produce it. She shares my belief in the spiral of life, where we come around to the same place where we were helped, but in a different, and hopefully higher, plane.

And just as we were helped in our youth, we have an obligation to help the next generation. Such is the cycle of life.

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Differential Equations for High School Students

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I graduated from a small rural high school before attending a large state university to study electrical engineering. I was a good student, but the transition to university studies and higher mathematics was difficult for me. I had taken all the algebra classes offered in high school, plus trigonometry, but had barely even heard of calculus before entering college. But there I was, a freshman in college, taking a five-hour calculus course. And shortly after that, I was in my first electrical engineering course, doing circuit analysis and solving differential equations.

Although I managed to make it through both calculus and circuit analysis, I felt like I was always trying to catch up, not really understanding the mathematics but just going through the process. I know that I could have achieved much better grades in the classes if I had had the proper foundation before taking those classes.

This paper attempts to give you that background – it's the paper I wish I had had before I went to college. It assumes that you know basic algebra and trigonometry but does not require any knowledge of calculus. It is not a textbook and there are no problems for you to solve – you'll get lots of those later in your university classes :-). The intent of this paper is to give you a conceptual understanding of the elements that go into solving a differential equation. If you understand the concepts presented here, the university lectures will make a lot more sense to you. You'll still have to do a lot of work for the classes but, hopefully, you'll understand the material and how things work.

As a favor to me, I'd appreciate if you would send me an e-mail (at the address above) with nothing but a subject line of "Differential Equations" so that I can determine how many people are reading this paper. If you'd like to make comments, corrections, or offer suggestions about the paper, that would be appreciated, of course, but is not required.

There's a lot of background required to understand differential equations. While I don't address these subjects in a nice linear order, some of the areas I'll describe are the natural number *e* and complex numbers. I'll also attempt to describe some of the basic concepts of differential calculus and the relationship between the exponential function and the trigonometric functions.

This is a wide spectrum of information but it's all required for a full understanding of how to solve differential equations. And if you find that I missed something or did not explain things sufficiently for you to understand it, please let me know so I can improve future revisions of this paper.

The approach I take in this paper is not the same as most mathematics books. I'm a technical person but I'm also very visually oriented. When I think about a problem, I visualize the system in my mind – how it all fits together and interworks. I take the same approach in this paper. I want to present the information to you so that you can visualize how it all fits together. This means that I may not tell you everything about the subject I'm describing. I only tell you enough so that you can understand how to solve the problem. Later, when you learn more of the details, you can graft those details onto your vision of how things work and keep the details straight. If I gave you all of the details now, you may have trouble keeping them straight and understanding what's needed to solve the problem at hand. So, for those of you who know all the details, forgive me for not exploring all the corners and crevices of the subject. I ignore for a good cause.

Compound Interest

It may seem strange to begin an explanation of differential equations with compound interest, but bear with me. I'll tie it all together.

Let's say you put \$100 in the bank at 10% per year interest. At the end of a year, your account will be worth \$110. To calculate the value at the end of one year, we use the following equation.

$$y = 100 + (0.1 * 100)$$

Now factor out the 100

$$y = 100 * (1+0.1)$$

Now, instead of using \$100 and 10%, let us use *D* to represent the amount of money deposited and *i* as the interest rate per year and substitute in the last equation.

$$y = D * (1 + i)$$

With this general equation, we can calculate the amount of money we will have after one year, for any deposit and interest rate. Now, let's take a longer look. How much money will you have in the account after two years? Let's take our original numbers and do the calculations. We know that after one year we'll have \$110. Substituting that into our general equation, along with 10% for the interest rate, this is what we come up with.

$$y_2 = 110 * (1 + 0.1)$$

$$y_2 = 110 * 1.1$$

$$y_2 = 121$$

Let's try to look at this in a more general way. The amount of money in the bank after one year is 100*1.1. The amount of money after two years is that amount of money, 100*1.1 times the 1.1 for the second year. Let's put this in an equation.

$$y_2 = (100 * 1.1) * 1.1$$

$$y_2 = 100 * 1.1 * 1.1$$

$$y_2 = 100 * (1.1)^2$$

In the general case, the equation for the amount of money which will be in the bank account after *n* years, if the interest is compounded annually is

$$y = D * (1+i)^n$$

Where *D* is the amount of money originally put on deposit and *i* is the interest rate per year. There's a trick that's used in compound interest tables and that is to make *D*, the amount of money originally deposited, equal to \$1. This allows us to simplify the equation to

$$y = (1+i)^n$$

Thus, if we have some money on deposit and want to know what its value will be after a certain number of years, we can look up the factor $(1+i)^n$ in a table, or calculate it on a handheld calculator, and then multiple the amount of money on deposit by that factor.

Suppose another bank wants our business and wants to make us a better offer. One way would be to offer us a higher interest rate. But there's an alternate way, and that's to compound the interest more often than once per year. Let's start by assuming that they agree to compound twice a year (biannually).

Let's begin by looking at how to calculate the biannual interest for one year. Again, we start with \$100 and 10% interest per year but we will compound it after six months instead of a year. How do we do that?

Since the period is half the annual period, we divide the interest rate by 2 and do the calculations just as before. The amount of money in the account after six months is

$$y = 100 * (1 + .1/2)$$

$$y = 100 * (1 + .05)$$

$$y = 100 * (1.05)$$

$$y = 105$$

Then, we use this amount as the basis for the calculation for the second six months.

$$y = 105 * (1.05)$$

$$y = 110.25$$

When we compounded only once per year, the account grew to \$110 at the end of one year. With biannual compounding, the amount has grown to \$110.25. So compounding more often within the year grows our money faster. Let's examine this in more detail. First, let's determine the general equation for compounding more than once per year. Let's use the variable *x* to represent the number of compounding periods in a year. The general equation for the amount of money in the account, given a starting amount of *D*, an annual interest rate of *i*, compounding *x* times in a year, and *n* years is

$$y = D\left(1 + \frac{i}{x}\right)^{xn}$$

Let's simply this a bit by investigating what happens in one year as we compound more and more times within the year. So in this case, n = 1, giving the equation.

$$y = D\left(1 + \frac{i}{x}\right)^x$$

Now, make D = 1 so that we can simplify further.

$$y = \left(1 + \frac{i}{x}\right)^x$$

To do the calculations, I'm going to do one more simplifying thing and that is to make *i* = 1 (100% interest), leading to the following equation.

$$y = \left(1 + \frac{1}{x}\right)^x$$

It's easy to see that if we compounded only once per year, the amount of money in the account would double after one year, to $2 in our example with 100% interest, since we assumed an initial deposit of $1. Now let us calculate the amount if we compound more often than once per year.

Period	**Value of x**	**Value of $(1+1/x)^x$**
Annually	1	2
Biannually	2	2.25
Quarterly	4	2.44140625
Monthly	12	2.61303529022
Weekly	52	2.69259695444
Daily	365	2.71456748202
Hourly	8,760	2.71812669162
Each minute	525,600	2.71827924259
Each second	31,536,000	2.71828178130

Table 1: Value of $(1+1/x)^x$ as the number of compounding periods increases.

So, if we compound biannually, each dollar of deposit will be worth $2.25 after one year. If we compound monthly, each dollar of deposit will be worth $2.61 after one year, or $0.36 more than biannual compounding. Note, also, that as we compound more and more often, the value of each dollar of deposit grows more slowly – if we compound hourly, the value of each deposit only grows to $2.72, or $0.11 more than monthly compounding. Beyond hourly compounding, there's very little growth in the value of the deposit.

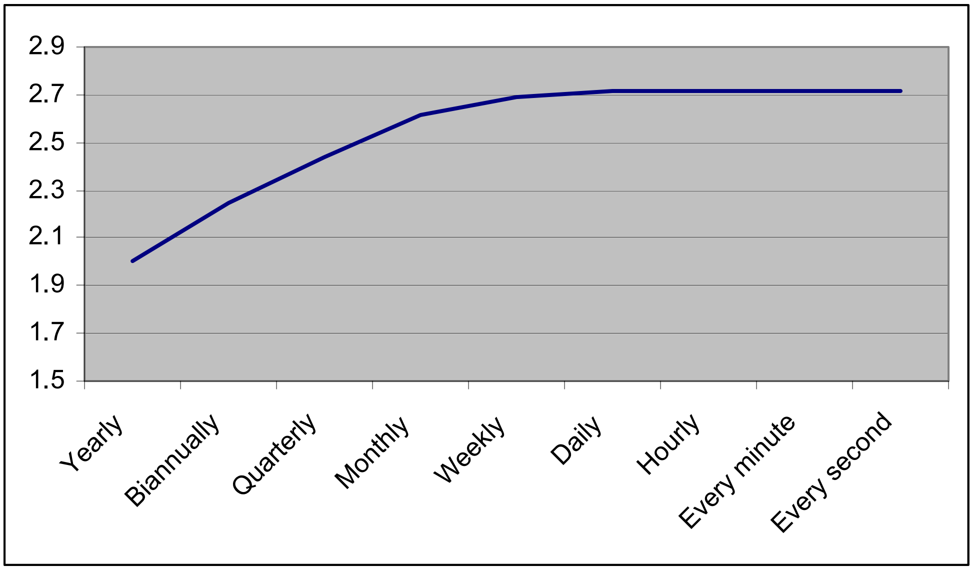

Figure 1: A plot of the values in Table 1. Notice how they appear to be asymptotic to some value.

Note how the value of $\left(1+\frac{1}{x}\right)^x$ seems to be converging toward some value. As x gets larger and larger, the value of the equation $\left(1+\frac{1}{x}\right)^x$ converges to special number, which we call *e*, the exponential number. Mathematically, we write this as follows:

$$e = \lim_{x\to\infty}(1+\tfrac{1}{x})^x$$

Now, *e* is just a number, but like π (which is also just a number), it has some very important properties, a few of which we'll explore in later sections. But first, let's take a look at logarithms.

Computing e with an infinite series

e can also be computed with the infinite series

e = 1 + 1/1! + 1/2! + 1/3! + 1/4! + …
(4! = 4 * 3 * 2 * 1)

This is an excellent way to calculate the value of e quickly and accurately. The "compound interest" equation given in the text will usually give an incorrect result beyond 5 to 6 digits when calculated with a spreadsheet or calculator because of rounding errors and the large powers. When using an Excel spreadsheet, for example, thirteen terms of the above series gives the result 2.7182818284 which is accurate to 10 decimal places.

Sixteen terms produces an answer accurate to 13 decimal places. 2.7182818284590

Logarithms

It's difficult for us, with our personal computers and hand-held calculators, to understand what a tremendous achievement the development of logarithms really was. Prior to logarithms, calculations were extremely laborious, especially for people such as astronomers who needed to multiply and divide very large numbers. With logarithms, multiplication of two numbers becomes an addition, and division becomes a subtraction (as will be explained later in this section). Addition and subtraction are much easier than multiplication and division, and less prone to error.

Even after logarithms, calculations were tedious but they were a lot less so. Logarithms were commonly used until at least the mid 1970's when low cost hand held calculators became popular.

One common way of using logarithms was in the form of a slide rule, an accessory which unmistakably identified the university engineering students. Figure 2 and Figure 3 are pictures of the author's slide rule, used during his university studies.

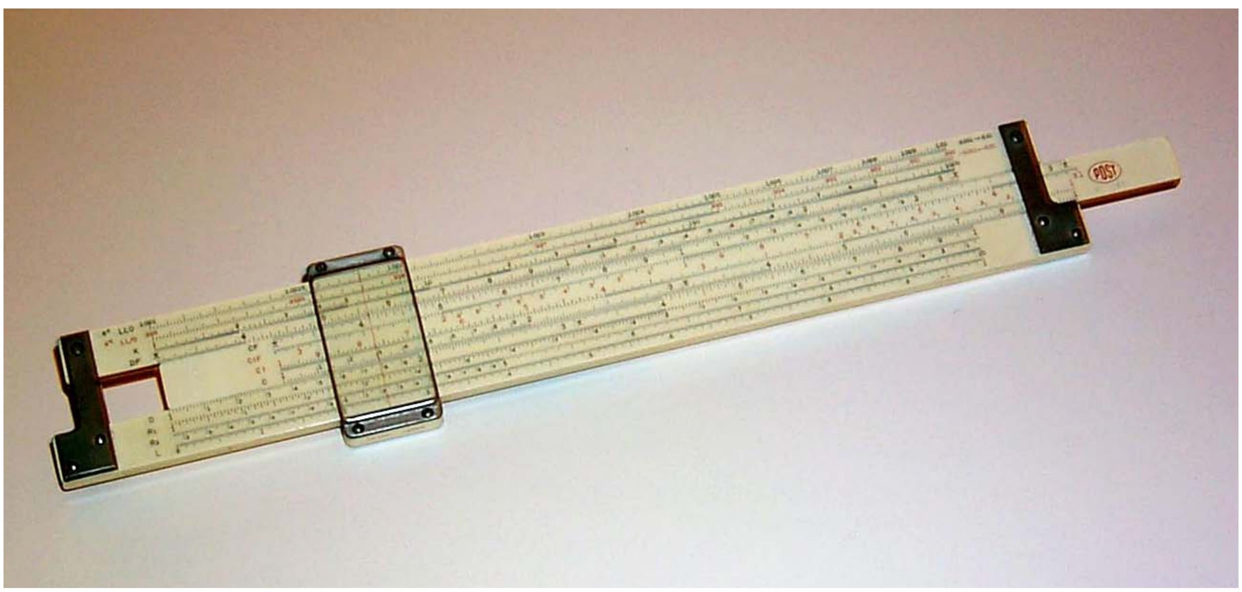

Figure 2: A slide rule uses logarithmic scales to perform multiplication and division. Prior to the availability of low cost calculators, this was the most popular calculating tool used by engineers and scientists.

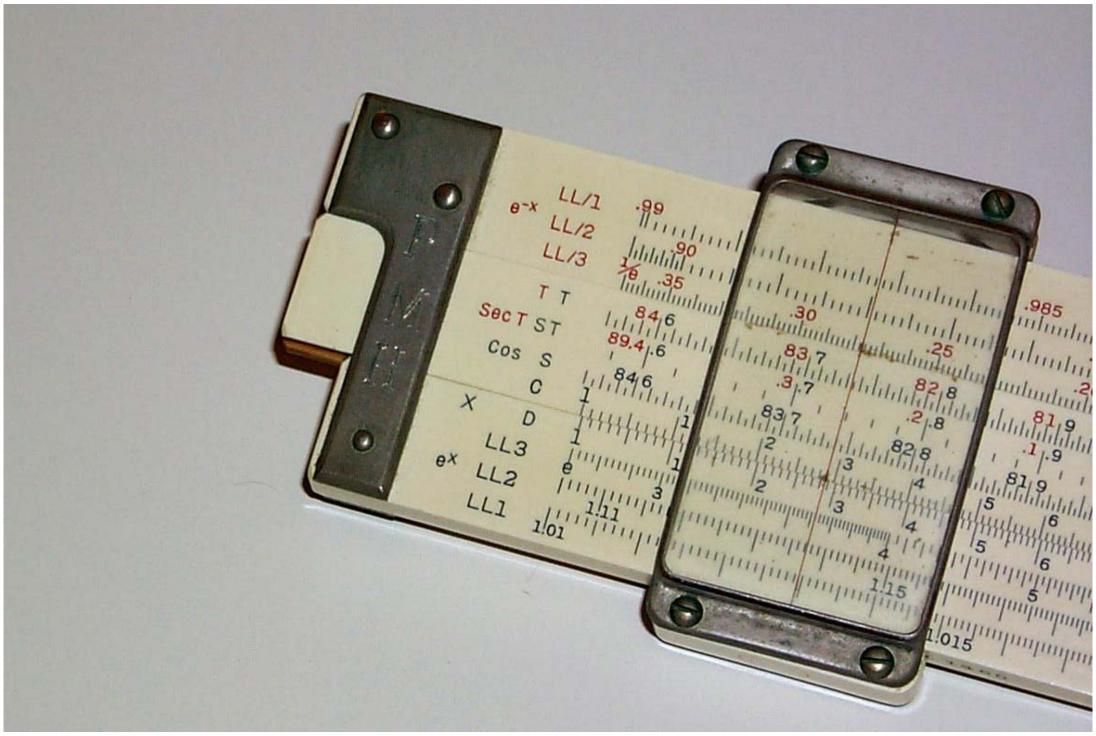

Figure 3: Most engineering students used the same brand of slide rule. But each student adjusted their slide rule differently, some with the slide very loose while others liked the slide tight. When working together on a problem, sometimes we'd lose track of which slide rule was which. To identify my slide rule, I had my initials (PMH) engraved on the rule. I was a real geek – and maybe I still am!

In discussing logarithms, I'm going to first discuss how logarithms work. After that, I'll discuss natural logarithms, based on the exponential number *e*.

Working with Logarithms

The basic insight that makes working with logarithms interesting is that when you multiple or divide numbers expressed as powers of a common base, you can just add or subtract the exponents. Let's examine this in more detail. Let's say that we have some numbers that are powers of 2, such as 4 and 8. 4 can be expressed as 2^2 and 8 can be expressed as 2^3. If we want to multiply these numbers, we can express this as

$$y = 4 * 8 = 2^2 * 2^3$$

We know from the law of exponents that when we multiply two numbers which have the same base, we just add the exponents.

$$y = 4 * 8 = 2^{2+3} = 2^5 = 32$$

If we were to divide 4 by 8, we'd subtract the exponents.

$$y = 4/8 = 2^2 / 2^3 = 2^{2-3} = 2^{-1} = 1/2$$

In the general case, we can express this as

$$y = b^x * b^z = b^{x+z}$$

and

$$y = b^x/b^z = b^{x-z}$$

So how do we express all this in logarithms? Let's go back to our original example where we used powers of 2, and look at the number 8.

$$y = 8 = 2^3$$

If we were to take the log to the base 2 of both sides, this is what we'd get.

$$\log_2 y = \log_2 8 = \log_2 2^3$$

By taking the log to the base 2, we're asking, "What power do we have to raise 2 to to equal 8?" We know the answer is 3 but let's look at the algebraic manipulations to get to that answer. One "rule" of logarithms is that the log of a number to a power is equal to the power times the log. So this gives us

$$\log_2 8 = 3 \log_2 2$$

Our task now is to evaluate $\log_2 2$. The meaning of this is, "What power must 2 be raised to to equal 2?" The answer is 1 because $2^1 = 2$. So our result is

$$\log_2 8 = 3$$

There's something very important that needs to be pointed out here. To a very large degree, *it doesn't matter what the base b is*. Obviously, *b* cannot be zero or one. A practical system requires that *b* be a positive real number greater than 1. One of the most common values of *b* is 10. Now, let us explore common (base ten) logarithms a bit more.

The definition of a common logarithm is

$$y \equiv 10^{\log y}$$

That is, the common logarithm of a number is the power that 10 must be raised to to equal that number. The logarithm of 10 is 1 because $10^1 = 10$. The logarithm of 2 is about 0.301 because $10^{0.301} = 2$ (approximately).

The advantage of working with logarithms to the base 10 is that we can easily calculate the logarithms of larger and smaller numbers, as explained in the box to the right.

Logarithms of larger and smaller numbers

Once we have the log of 2, we can use that result to obtain the logarithms of larger numbers. For example, suppose we wanted to take the log of 20. 20 can be expressed as

$20 = 10 * 2$

or as powers of 10

$20 = 10^1 * 10^{0.30102}$

$20 = 10^{1+0.30102}$

$20 = 10^{1.30102}$

So the log of 20 is 1.30102…

This is the primary advantage of logarithms to the base 10. They allow you to easily find the logarithms of numbers larger than 10 or smaller than 1.

But logarithms do not have to be based on the number 10 – they can be based on almost any number. And later, we'll see that logarithms to the base *e* have certain advantages.

Before leaving logarithms, there's one more thing I need to discuss, and that's how to convert from logarithms of one base to logarithms of another base. For example, suppose we had logarithms to the base 10 and we wanted to convert them to logarithms to the base *e*. Can we do that? And of so, how?

As stated above, the definition of a logarithm is

$$c = b^{\log c}$$

If you have trouble with variables, think of *b* as being 10. Now, we want to compute the log of *c* to the base *x*. If you want to, think of *x* as being the value *e*. Or,

$$c = x^{\log c}$$

Let's take the log to the base *b* of both sides.

$$\log_b c = \log_b (x^{\log c})$$

Since the log of a number to a power is equal to the power times the log,

$$\log_b c = \log_x c * \log_b x$$

Since what we want is the log of *c* to the base *x*, we solve for that.

$$\log_x c = (\log_b c)/\log_b x$$

The answer is that we divide all our existing base *b* logarithms by $\log_b$ x. So to convert logarithms of the base 10 to logarithms to the base *e*, we divide each of our logarithms to the base 10 by $\log_{10}$ *e*.

Now, we're going to move on to a very interesting subject, complex numbers.

Complex Numbers

Up to this point in school, the kinds of equations presented to you have been carefully controlled. For example, unless you've studied imaginary numbers, you probably never encountered an equation like the following.

$$x^2 + 1 = 0$$

The reason you were never presented such an equation is that it has a very special solution, or roots. Let's solve this equation.

$$x^2 = -1$$

$$x = \pm\sqrt{-1}$$

But there is no number which when squared is equal to –1. We need to solve these equations so we shall invent such a number and call it *i*, the imaginary number. The definition of *i* is

$$i^2 = -1$$

Note that the solution of our original equation

$$x^2 + 1 = 0$$

is actually

$$x = \pm i$$

Minus *i* is just as valid a solutions as plus *i*. It turns out that we can use either $+i$ or $-i$ in our equations and we'll get the same result – but you must be consistent. If you choose to use $-i$ you must do so everywhere. If you use $+i$ sometimes and $-i$ other times you'll probably get errors.

Given this, we can use *i* in our mathematics and apply all the rules which we have learned over twelve years of school, and all those rules will still make sense when *i* is utilized. In fact, we find that numbers are now expressed as $(a + ib)$, where a and b are real numbers. The whole number $(a + ib)$ is called a complex number. The first part, a, is considered the real part of the complex number, while the remaining part, ib, is considered the imaginary part. You've been working with complex numbers in all of your mathematics classes, but the imaginary part has been equal to zero ($b = 0$).

Let's look at some arithmetic with complex numbers.

$x = 3 + i4$

$y = 5 + i6$

To add the two numbers, we simply add the real and imaginary parts separately. This gives the result

$x + y = (3 + i4) + (5 + i6)$

$x + y = 3 + 5 + i4 + i6$

$x + y = 8 + i10$

Subtraction is exactly the same as addition with a sign change.

$x - y = (3 + i4) - (5 + i6)$

$x - y = 3 + i4 - 5 - i6$

$x - y = 3 - 5 + i4 - i6$

$x - y = -2 - i2$

Multiplication is a bit more complex. Let's look at the general case first

$x = a + ib$

$y = c + id$

$x * y = (a + ib) * (c + id)$

$x * y = ac + icb + iad + i^2db$

Since $i^2 = -1$,

$x * y = ac + icb + iad + db*(-1)$

$x * y = ac - db + i(cb + ad)$

Going back to our original example

$x * y = (3 + i4) * (5 + i6)$

$x * y = 15 + i20 + i18 + i^2 24$

$x * y = 15 - 24 + i\,20 + i18$

$x * y = -9 + i38$

Division is a problem because we don't know how to divide by *i*. A mathematical "trick" is used to avoid having to divide *i* by *i*. That trick is to multiply the denominator by its conjugate. The conjugate of a complex number is the same number, with the sign of the imaginary term reversed. For example, the conjugate of 5 + 6i is 5 – 6i. When a complex number is multiplied by its conjugate, the imaginary terms all disappear, leaving only the real term. This allows us to divide by a real number and sidesteps the question of how to divide by an imaginary term. Let's do some division.

$$\frac{x}{y} = \frac{3+i4}{5+i6}$$

Since we want to multiply the denominator by its conjugate, we must multiply the numerator by the same value so that the net effect of the multiplication is multiplication by 1.

$$\frac{x}{y} = \frac{3+i4}{5+i6} * \frac{5-i6}{5-i6}$$

$$\frac{x}{y} = \frac{(3+i4)*(5-i6)}{(5+i6)*(5-i6)}$$

Expanding

$$\frac{x}{y} = \frac{15+i20-i18+24}{25+i30-i30+36}$$

$$\frac{x}{y} = \frac{39+i2}{61}$$

$$\frac{x}{y} = \frac{39}{61} + \frac{i2}{61}$$

This takes care of addition, subtraction, multiplication, and division of complex numbers. The one operation which we have not yet covered is exponentiation, or the raising of a number to a complex power. This is perhaps the most difficult problem in complex numbers.

When dealing with real numbers, we were told that raising a number to a power was equivalent to multiplying the number by itself the exponent number of times. For example b^3 means to multiply b by itself three times (b * b * b). Likewise, you can understand $b^{1/3}$ by recognizing that you must raise this number to the third power $(b^{1/3})^3$ to achieve the result b^1, and therefore, $b^{1/3}$ must mean the cube root of b.

Negative exponents can be understood by recognizing that exponents are subtracted when the base numbers are divided. For example, b^{-3} can be interpreted as b^{0-3} or b^0/b^3, or $1/b^3$. Thus, we can easily understand how to deal with all types of real exponents.

But when we have a complex exponent, we cannot directly compute the value of the number. Again, we must find a "trick" to be able to find the value of a number raised to a complex power, such as b^{i3}.

You probably won't be surprised to learn that the trick involves the number e. The first thing we note is that any real number b can be represented by e to a constant power, e^k. After all, e is just a number. So

b^{i3} can be represented as $(e^{k})^{i3}$ or e^{i3k}. So if we can find a way to evaluate e^{ix}, we can evaluate any real number to an imaginary power.

To evaluate e^{ix}, we first look at the infinite series that can be used to calculate e^{x}, where x is a real number (not an imaginary number). The reason this infinite series converges to e^{x} will not be explained here – it requires knowledge of more calculus than is assumed in this paper. For now, take my word for it[1].

$$e^{x} = 1 + x^1/1! + x^2/2! + x^3/3! + x^4/4! + x^5/5! \ldots$$ (5! is read "five factorial" and means 5*4*3*2*1)

It turns out that this series is still correct when we substitute ix for x.

$$e^{ix} = 1 + (ix)^1/1! + (ix)^2/2! + (ix)^3/3! + (ix)^4/4! + (ix)^5/5! \ldots$$

But note that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, etc. Using that knowledge, we can rewrite the series as

$$e^{ix} = 1 + ix^1/1! - x^2/2! - ix^3/3! + x^4/4! + ix^5/5! \ldots$$

Let's do some simple algebraic manipulations on this series, collecting all the terms without i in one place, and all the terms with i in a different place.

$$e^{ix} = (1 - x^2/2! + x^4/4! - x^6/6! + \ldots) + (ix^1/1! - ix^3/3! + ix^5/5! - ix^7/7! + \ldots)$$

We can now factor the i term from the second group of fractions.

$$e^{ix} = (1 - x^2/2! + x^4/4! - x^6/6! + \ldots) + i(x^1/1! - x^3/3! + x^5/5! - x^7/7! + \ldots)$$

Now, the trigonometric function cosine has an infinite series expansion of

$$\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + \ldots$$

And the sine function has an infinite series expansion of

$$\sin x = x^1/1! - x^3/3! + x^5/5! - x^7/7! + \ldots$$

Substituting into the equation for e^{ix} gives

$$e^{ix} = \cos x + i \sin x$$

Although it's probably not apparent to you right now, this is one of the most beautiful and wonderful equations in mathematics. It relates an algebraic function, e^{ix}, to trigonometric functions, specifically the cosine and sine. It's commonly known as Euler's Equation[2] (and sometimes Euler's Identity). In science and engineering we often have to deal with oscillations. For example, a large bridge or building will sway in the wind. In fact, it oscillates. Without the above equation, we would have to analyze this motion using trigonometric functions, and doing algebra with trigonometric functions is quite difficult. Using the above equation, and some related equations which I'll give you below, we can analyze the motion using the ordinary algebraic functions of addition, subtraction, multiplication, division, and exponentiation.

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[1] You'll learn more about this when you study Taylor and MacLauren series in advanced calculus.
[2] Leonhard Euler, (1707 – 1783), Swiss mathematician.

Now, here are a few related equations.

$$e^{-ix} = \cos x - i \sin x$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

There's one more thing I need to discuss before leaving this subject. In all of your higher mathematics courses, angles will be measured in radians instead of degrees. There are 2π radians in a circle.

In your trigonometry courses you probably measured angles in degrees, with 360 degrees in a circle. But there's nothing special about measuring angles in degrees. The reason we have 360 degrees in a circle is simply a quirk of history, because of the Babylonian base 60 numerical system. We could just have easily defined a circle to be 100 degrees and everything would still work fine. In higher math, we simply define a circle to consist of 2π radians. So 180 degrees = π radians, 90 degrees = π/2 radians, etc.

You may ask, "Why use radians instead of degrees?" Well, one reason is that π is a basic relationship for circles. For example, the circumference of a circle is π times the diameter, or π times twice the radius ($C = 2\pi r$). The area of a circle is πr^2.

We could certainly do all the calculations with 360 instead of 2π, and we'd still get useful, valid answers. But when we use radians, answers work out to be "neater" – we get answers with factors of π in the answers instead of 180 degrees. One interesting thing to note is that there are not an "even" number of radians in a circle, since π is an irrational number (it has a never ending, non-repeating decimal fraction).

Now in reality, most people who work with radians still think in degrees. I still convert radians to degrees in my head when I need to know the value of a sine or cosine. My high school trigonometry class is still with me! Figure 4 below shows the relationship between degrees and radians for many common angles.

But my practical advice to you is, "Get used to radians. You're going to be using them a lot."

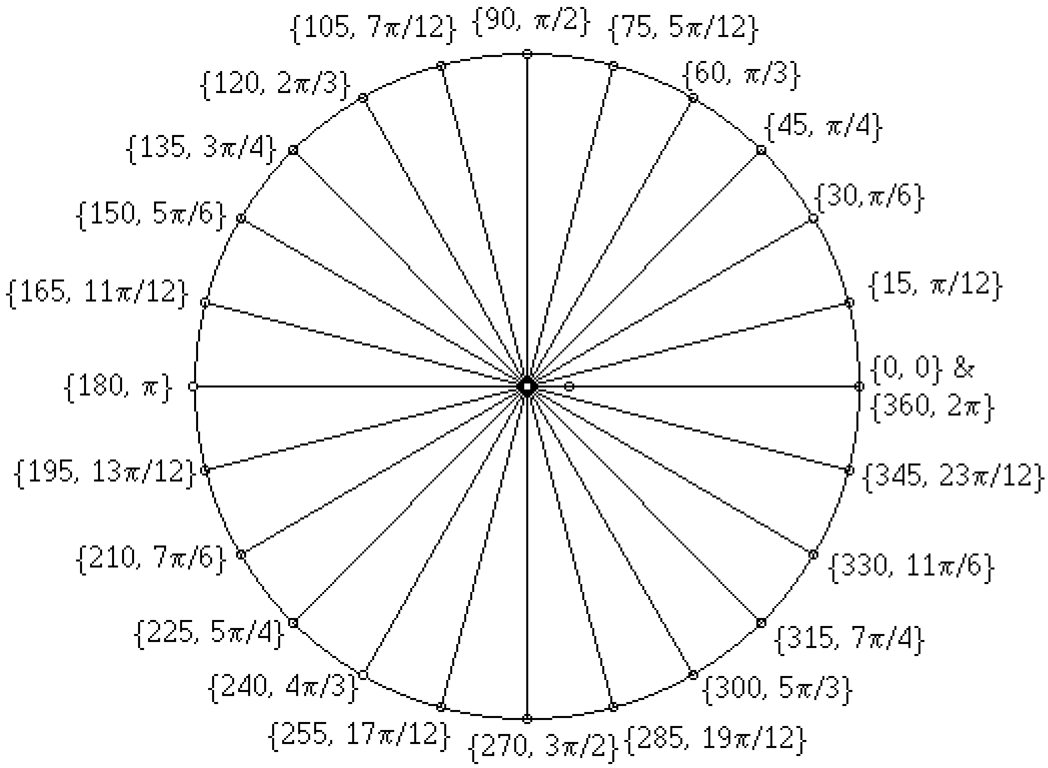

Figure 4: The relationship between degrees and radians for common angles.

So, now that we understand radians, let's look at a special case of Euler's equation – its value when $x = \pi$ (or 180 degrees).

$$e^{i\pi} = \cos \pi + i \sin \pi$$

It turns out that $\cos \pi = -1$ and $\sin \pi = 0$, so

$$e^{i\pi} = -1 \qquad \text{(this is sometimes written as } e^{i\pi} + 1 = 0\text{)}$$

People have made a great deal about this result because it brings together *e*, *i*, and π for an integer result of –1. But it's just a fluke of the mathematics. Here, π represents a half rotation (180 degrees). At that point, the cosine is equal to –1 and the sine to zero (which makes the imaginary part disappear). Note that there's two other places where the sine is equal to zero, and that's at zero radians (zero degrees) and at 2π radians (360 degrees). In both cases, the cosine is equal to +1. For zero radians, the equation would be

$$e^{i0} = 1$$

No one makes a big deal out of this equation because any number to the zero power[3] is 1. But if you want to get mystical, you can say that this equation relates *e*, *i*, and 0 to give a positive real number.

For 2π radians, we get the same result.

[3] Except zero to the zero power, which is indeterminate.

$$e^{i2\pi} = +1$$

Which is equivalent to squaring our original equation,

$$e^{i\pi} = -1$$

$$(e^{i\pi})^2 = (-1)^2$$

or

$$e^{i2\pi} = +1$$

Another "interesting" result is obtained by substituting $x = \pi/2$ (90 degrees). In this case, $\cos x = 0$ and $\sin x = 1$, leading to $e^{i\pi/2} = i$. Now, let's play around a little bit and take the square root of this equation.

$$(e^{i\pi/2})^{1/2} = i^{1/2}$$

$$\sqrt{i} = \pm e^{i\pi/4} = \pm(\cos \pi/4 + i \sin \pi/4) \qquad \text{(note: } \pi/4 = 45 \text{ degrees)}$$

Note that both the cos and sin of $\pi/4$ is $1/\sqrt{2}$. So the square root of i is

$$\sqrt{i} = \pm e^{i\pi/4} = \pm\frac{1}{\sqrt{2}}(1+i)$$

It's easy to get confused by the "strange" results you can get by manipulating the e^{ix} equation. For example, in this last example you may be thinking, "How can I take the square root of the square root of -1?"

I have two answers for you. First, you'll see in the next section that complex numbers have a geometrical interpretation, and the results must be viewed in light of this interpretation. Second, on a more personal level, don't try to be literal all the time. Recognize that mathematics deals with abstractions and work with the abstractions. If you're careful to apply the mathematics properly, everything will work out and you'll get useful results[4].

Complex numbers have an important role in the solution of equations with powers of x greater than 2. For example, what's the cube root of 8? You'll quickly answer "Two", but the fundamental theorem of algebra says that for an equation of degree n (and a cubic equation is degree 3), there are n real or complex roots. One of the cube roots of 8 is indeed 2, but there are two other roots which are complex[5]. So learn complex number theory – complex numbers show up in many places in advanced math.

[4] For another "strange" result, use the equation for i we just developed ($e^{i\pi/2} = i$), but instead of taking the square root of i, raise i to the i power. You'll find that the result is *not* a complex number – it's a real number. The math works out fine but try to visualize it! Also, using this equation ($e^{i\pi/2} = i$), you can take the natural log of both sides, giving $\ln i = i\pi/2$. Can you see why? If you have trouble, see Appendix B.

[5] The three cube roots of 8 are 2, $-1+i\sqrt{3}$, and $-1-i\sqrt{3}$. See Appendix B for a more complete discussion of this.

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Visualizing
$e^{ix} = \cos x + i \sin x$

So now that we have this beautiful and wonderful equation, what does it mean? Let's spend some more time examining $e^{ix} = \cos x + i \sin x$.

First, you'll notice that the equation

$$e^{ix} = \cos x + i \sin x$$

is just a standard complex number of the form

$$y = a + i\, b$$

where $y = e^{ix}$, $a = \cos x$ and $b = \sin x$. Since a is equal to the cosine, it will vary between +1 and –1, while b, the sine, will vary from 0 to +1, to 0, to –1, to 0. But what geometrical interpretation do we put on this.

We visualize complex numbers as existing on a complex number plane, with the x-axis representing the real number portion of a complex number and the y-axis representing the imaginary portion. See Figure 5.

Figure 5 also shows the value of $x = \pi/6$ (30 degrees) plotted on the complex number plane. The value of $x = \pi/6$ gives a complex number of $(0.866 + i0.5)$. Since this is a Cartesian coordinate system, the length of the vector from the origin to the plotted point is given by

$$r = (a^2 + b^2)^{1/2}$$

Because of the way the sine and cosine work, r will always be equal to 1 for our equation $e^{ix} = \cos x + i \sin x$. If you don't believe me, try a few different values for x and compute the sum of the squares.

Now, let's choose a few values for x and plot the resulting values of $a + ib$ on the complex number plane. The values we'll plot (multiples of 30 degrees) are shown in Table 2. The values are plotted in Figure 6.

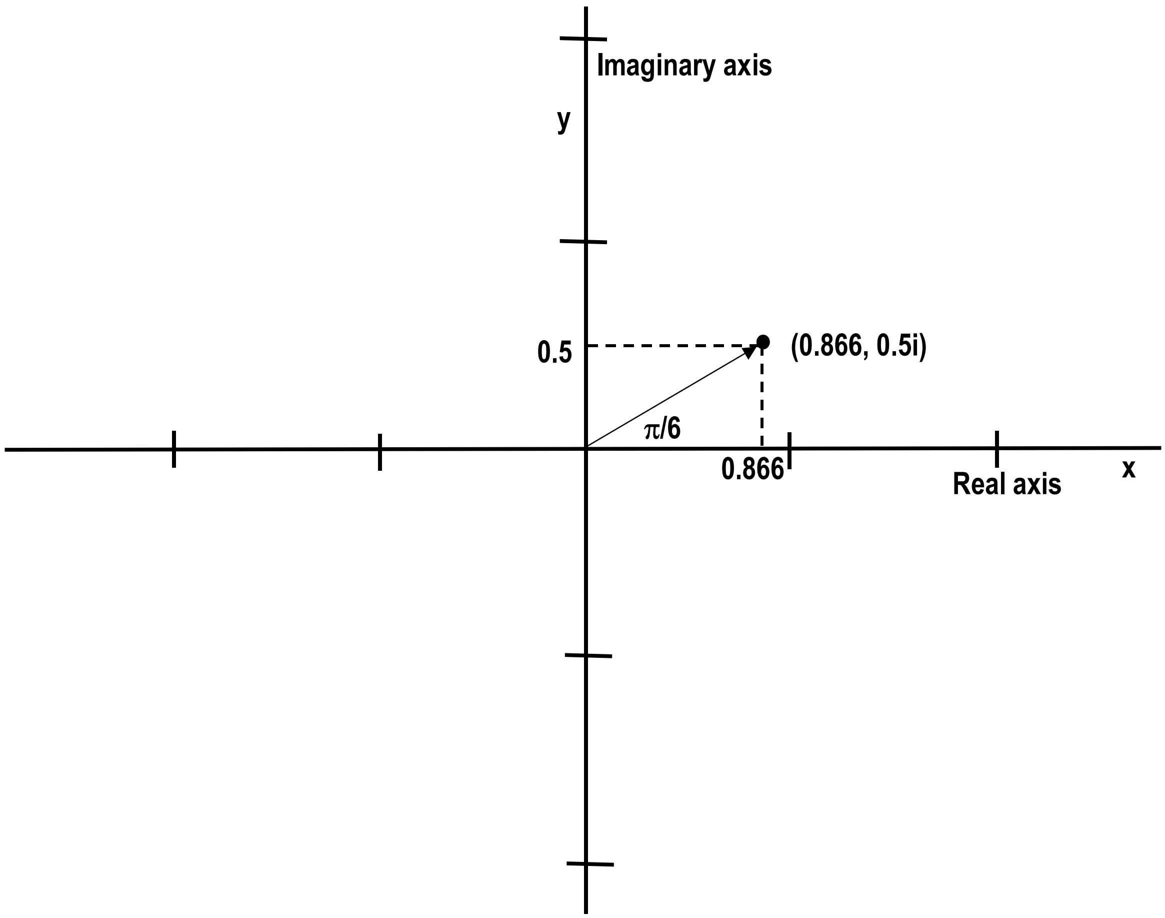

Figure 5: The complex number plane with the complex number 0.866 + *i*0.5 plotted.

Value of x	Real portion	Imaginary portion
0	1	0
π/6	0.866025404	0.5
π/3	0.5	0.866025404
π/2	0	1
2π/3	–0.5	0.866025404
5π/6	–0.866025404	0.5
π	–1	0
7π/6	–0.866025404	–0.5
4π/3	–0.5	–0.866025404
3π/2	0	–1
5π/2	0.5	–0.866025404
11π/6	0.866025404	–0.5
2π	1	0

Table 2: Value of real and imaginary parts of $e^{ix} = \cos x + i \sin x$ for different values of x. The values of x are multiples of 30 degrees.

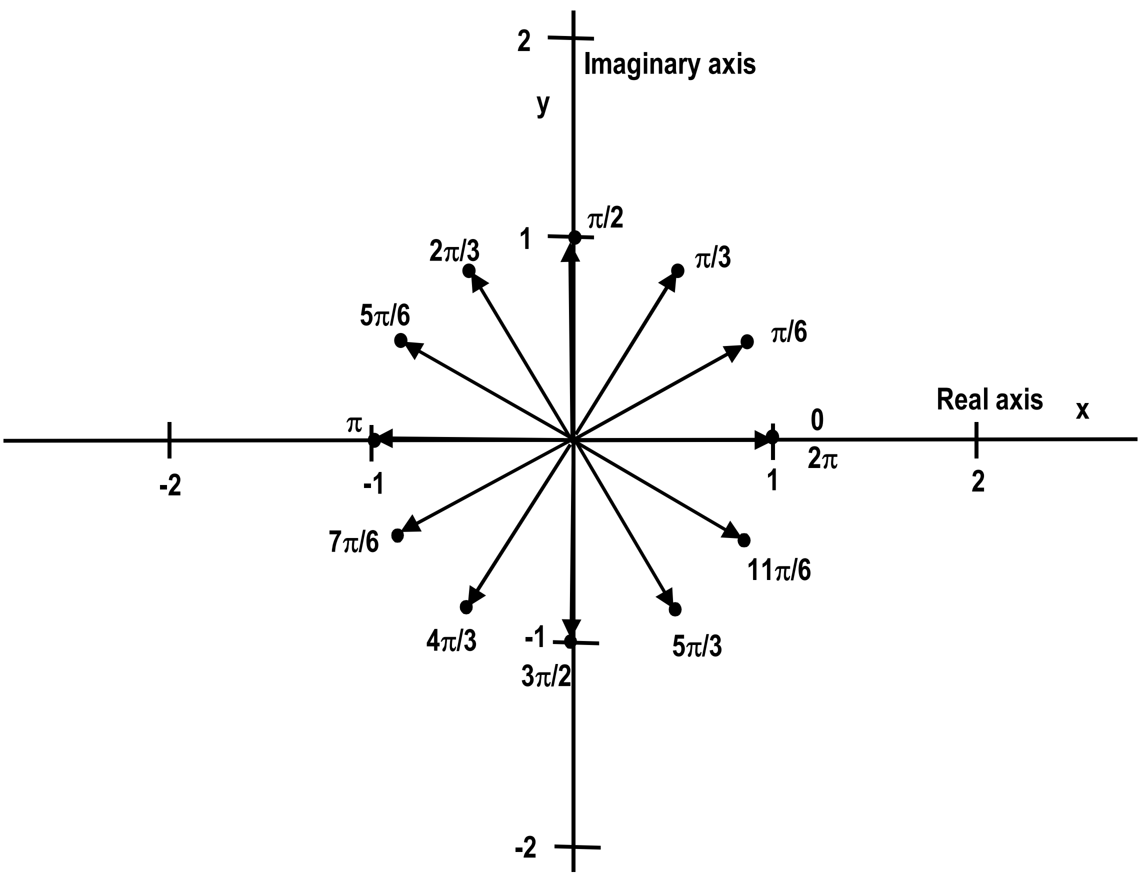

Figure 6: The complex number plane with the complex numbers shown in Table 2 plotted.

There are two things to note in Figure 6. First, the values lie on a circle of radius 1. Second, as the value of x increases, the radius vector moves *counterclockwise* around the origin.

Now, we've been looking at the location of the radius vector for a fixed value of x, and for each value of x that we chose, we computed the resulting location on the complex number plane. But suppose we had a computer which could automatically display the radius vector for any x. And suppose that we programmed that computer to increment the value of x by a small amount each second. What would we see?

We'd see the radius vector (an arrow with its tail at the origin) rotate around the origin in a counterclockwise direction. The speed of the rotation would depend upon how much we incremented the value of x each second. Of course, if we incremented x by too big a value, we'd see the arrow jump instead of moving in a smooth manner, but we could fix that by updating x more often, maybe 24 times a second and incrementing x each time by 1/24 of the value we had used previously.

That's one way to look at how to generate a rotating vector. But let me give you another technique. Suppose we replaced x with the value $2\pi t$, where t is time in seconds. Suppose we now let our computer increment t with the elapsed time, and we updated t 24 times per second. So each increment will be 1/24 of a second. We'd see the radius vector rotate smoothly around the origin, with each rotation taking one second.

This gives us a way to create a rotation of one rotation per second. This rotation is often called a "cycle", so here we have one cycle per second. Suppose we wanted to create rotations of more than one cycle per second. A way to do that is to add an addition value to our exponent, which was $2\pi t$. I'm going to choose the letter "*f*" to be added, and I'll explain later why I chose that letter. So our new exponent will be $2\pi ft$. If we want to have the radius vector rotate twice per second (two cycles per second) we make $f = 2$. If we want the radius vector to rotate five times per second, we make $f = 5$. And if we want the radius vector to rotate five million times per second, we let $f = 5{,}000{,}000$.

This notation is so common that there's a special symbol assigned to the sequence $2\pi f$. That variable is the lower case Greek symbol omega, ω. So now, what we have is $x = 2\pi ft = \omega t$. When written as an exponent of e, we have $e^{i\omega t}$. You'll see this notation over and over as you analyze oscillating systems.

Earlier I said that I'd explain why I chose "*f*" as the added variable. Since $e^{i\omega t}$ is commonly used to analyze oscillating systems, f is the frequency of oscillation. If you were analyzing a radio signal that operated at 54 megahertz, f would be the frequency – 54,000,000 hertz[6].

And just as a side note, note that $e^{-i\omega t}$ is equivalent to a vector rotating in the *clockwise* direction about the origin, since $e^{-ix} = \cos x - i \sin x$. If you go to Table 2 and change the sign of all the imaginary components, you can see that each vector will be "mirror imaged" around the x-axis.

So what good is all this? Although complex numbers can be handled mathematically, physical processes in the world are described by real numbers. Imaginary numbers do not describe actual physical processes. When we have a complex equation which describes a physical process, we ignore the imaginary part and only look to the real part for a description of the actual, measurable process. So why use equations with complex numbers? Because it's easier dealing with complex numbers than trying to do the mathematics with only real numbers, which in most cases means using trigonometric functions.

[6] In 1960, the term "cycles per second" was changed to "hertz" (abbreviated Hz) in honor of Heinrich Rudolf Hertz (1857 – 1894), a German physicist who applied Maxwell's theories to the production and reception of radio waves.

So how do we interpret the rotating vector described by $e^{i\omega t}$? We now know that only the real portion exists in the physical world, that is, the values on the x-axis. Let's see how we visualize this.

To do this, we'll take a short side trip to London. As London prepared for the millennium celebration, they built a huge Ferris wheel on the River Thames, close to Waterloo train station, called the London Eye. See Figure 7.

Looks like a big bicycle wheel, doesn't it? The wheel is gigantic – it's 450 feet high and there are 32 passenger capsules on its circumference. Look at the picture on the left. See the building behind the Ferris wheel[7]? Well, just beyond that building is a road that leads to Westminster Bridge over the River Thames.

If you were walking over Westminster Bridge, you'd come to a point where you were in line with the wheel – where you'd only see the passenger capsules going up and down and not around.

Now, suppose one of your American friends was on the Ferris wheel and had a US flag which he stuck out the side window of the capsule he was riding in. You'd see the flag going up and down, and since it was stuck out the side, you'd see it when the capsule was on the back side of the wheel as well as when the capsule was on the side closest to you. What would the motion of the flag look like?

[7] The building is the London Marriott Hotel Country Hall.

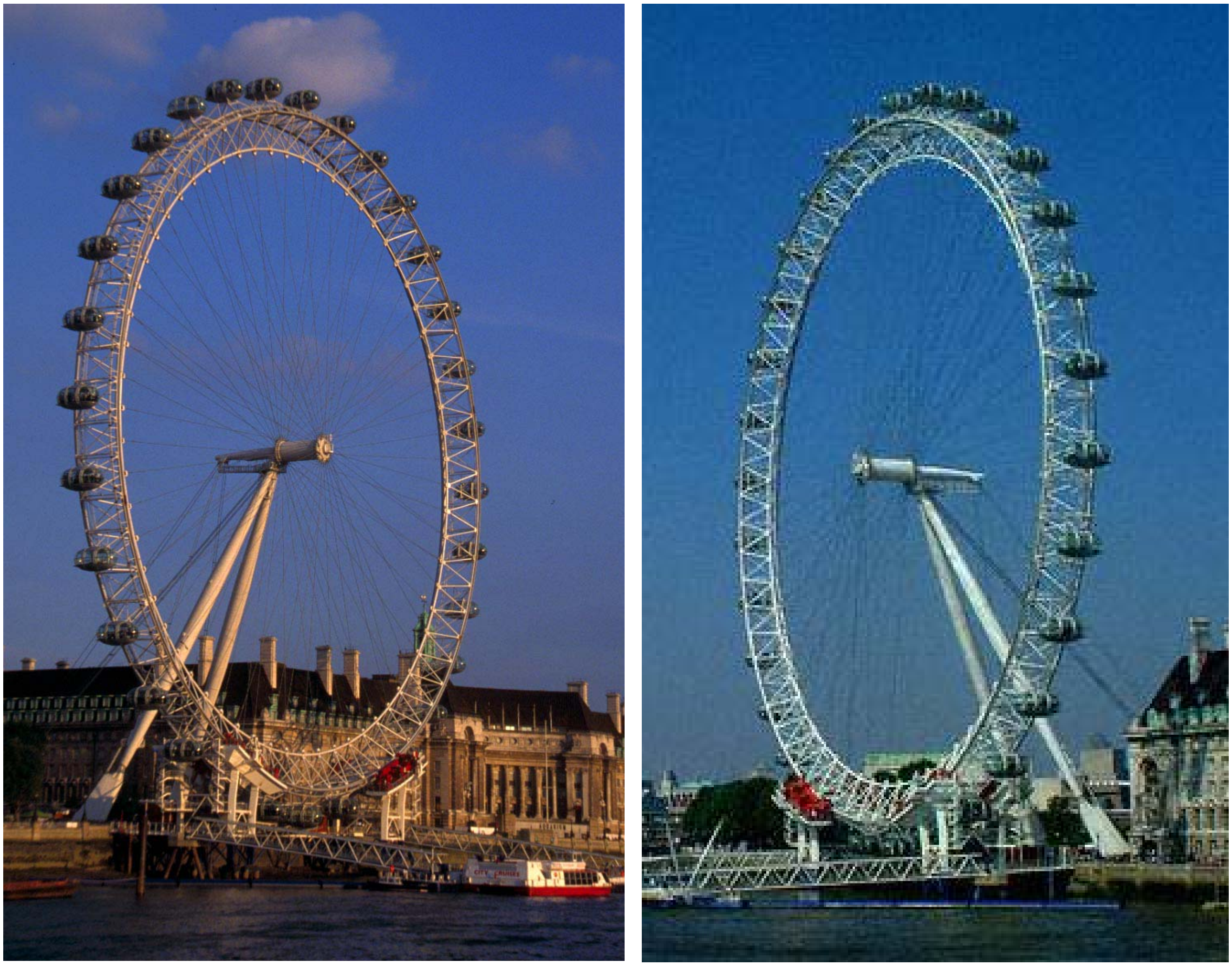

Figure 7: Two views of the London Eye Ferris wheel.

The flag would look like it was going up and down. Additionally, the flag would appear to be moving fastest when it was half way between the top and bottom of the wheel. The motion of the flag would be exactly the same as the value of real part of $e^{i\omega t}$ when ω is 2π times the rotational speed of the wheel.

Let's say that the wheel rotates once every six minutes. That's one rotation every 360 seconds, so $2\pi f$ is $2\pi/360$, or $\pi/180$. So the equation of motion, in our new complex notation is $e^{i\pi t/180}$ (in case you can't read the exponent, it's $i\pi t/180$).

Now, let's say that you had an optical measuring device which would tell you the height of the capsule at any time, and you began to plot the location of the flag every 30 seconds. You start when the capsule is at the top of the wheel. What will your plot look like? It'll look something like the plot in Figure 8.

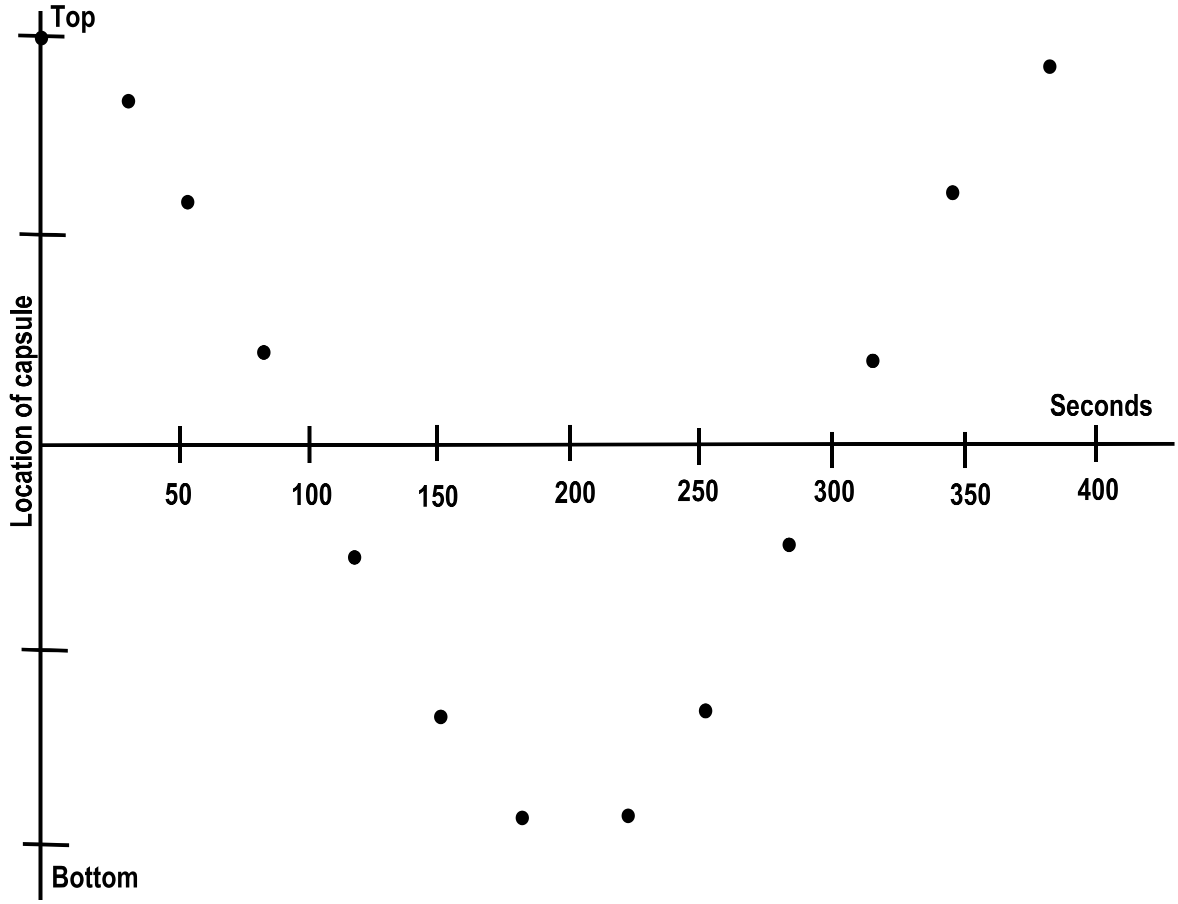

Figure 8: Plot of the location of the flag on the London Eye Ferris wheel.

Looks like a cosine curve doesn't it? The real part of $e^{i\omega t}$ describes a sinusoidal curve[8], which is exactly what we need to analyze oscillating motion.

So even though we took a trip through complex number land, we wind up with an analysis of sinusoidal (oscillating) motion in the physical world. We took our trip through complex number land because it was the shortest, easiest path to obtain the answers we need. This same sentiment was expressed by the French mathematician Jacques Hadamard (1865 – 1963): "The shortest path between two truths in the real domain passes through the complex domain[9]."

[8] Not a sine curve, but a sinusoidal curve. The cosine curve and the sine curve are both sinusoidal curves, meaning that the repeat in a periodic manner.
[9] Quoted in *The Mathematical Intelligencer* **13** (1991).

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Geometrical Implications of *i*

There's one more aspect of the imaginary number that I want to cover before we move on to differential equations, and that is the geometrical implications of multiplication by i and $-i$.

Let's begin by looking at a complex number on the complex plane. We'll take a simple number to make the plotting easier, using z = 2 + i3.

Let's multiply that number by $+i$.

$$z = i(2 + i3)$$

$$z = i2 + i^2 3$$

$$z = -3 + i2$$

Now, let's plot $2 + i3$ and $-3 + i2$. See Figure 9.

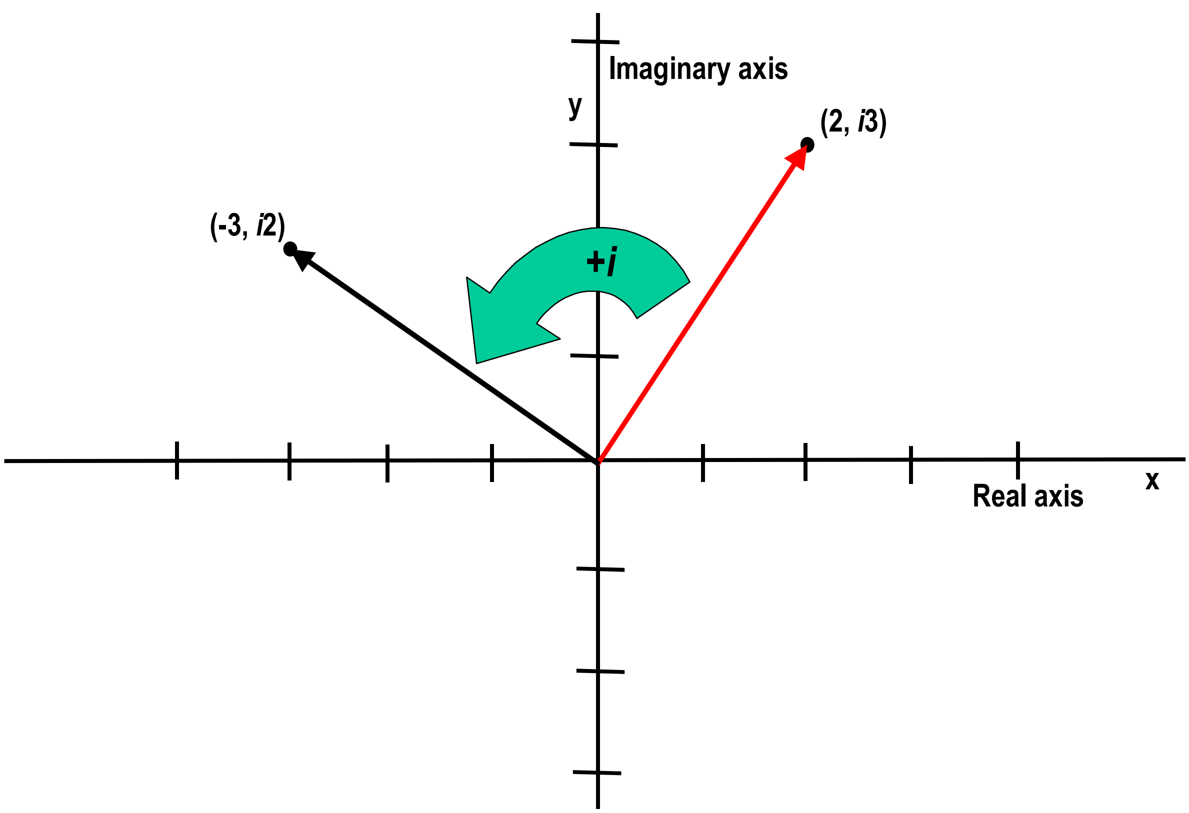

Figure 9: Rotation in the complex plane via multiplication by $+i$.

So we can see from the figure that multiplication by *i* causes the radius vector to rotate counterclockwise by 90 degrees (or $\pi/2$ radians). Now, let's examine multiplication by $-i$.

$z = -i(2 + i3)$

$z = -i2 + (-i)i3$

$z = +3 - i2$

So let's see what result we get if we now plot $2 + i3$ and $+3 - i2$. See Figure 10.

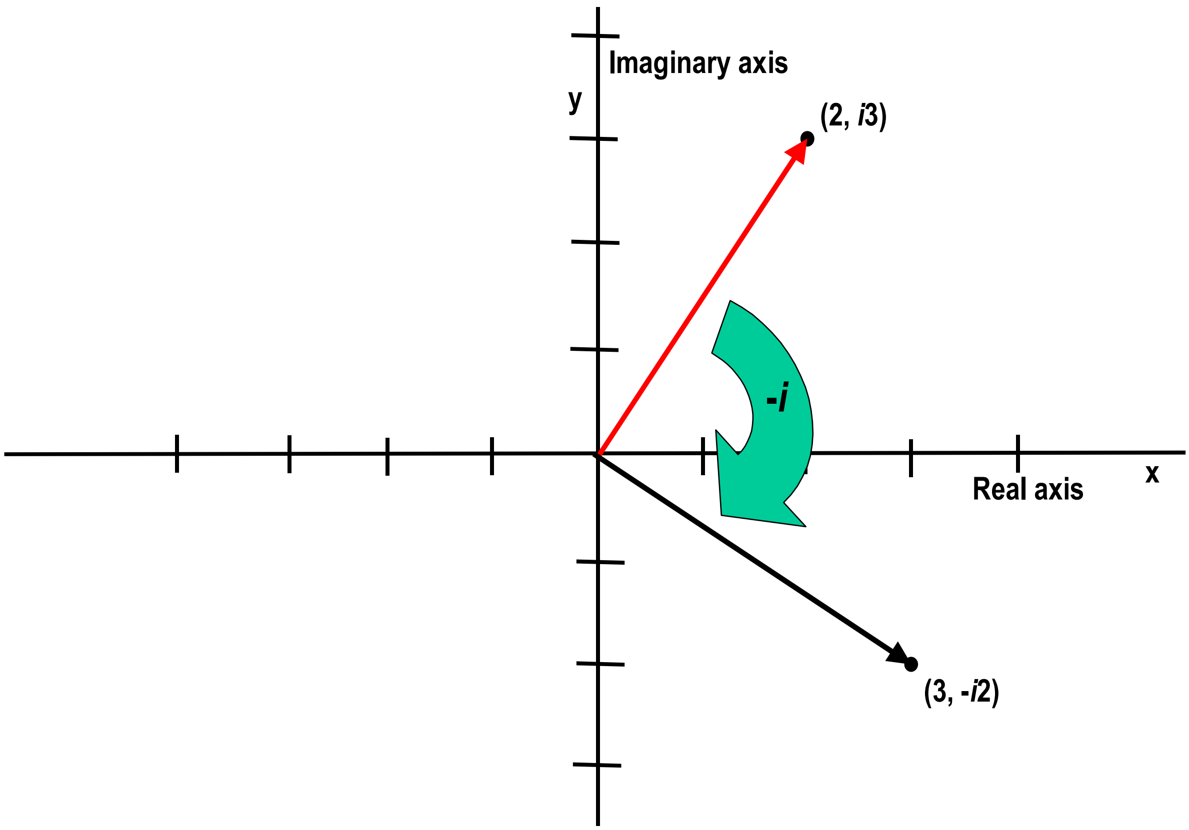

Figure 10: Rotation in the complex plane via multiplication by $-i$.

So multiplication by *i* causes rotation by 90 degrees in the counterclockwise direction, while multiplication by $-i$ causes rotation by 90 degrees in the clockwise direction. This is one of the most important concepts in complex numbers. If you don't remember anything else about complex numbers, remember this.

Differential Calculus

There are two major branches of calculus, differential calculus and integral calculus. In this paper, I will only address differential calculus.

When first exposed to calculus, it is not unreasonable for students to ask, “Why do we need calculus?” “What kind of problems are there that can’t be solved with algebra and need the tools of calculus?” There are a number of problems that need the tools of calculus but I’m going to concentrate on only one kind – and that is problems that ask you to find the maximum or minimum under certain conditions. Towards the end of this section, I’ll illustrate this by solving a problem to discover why tin cans are shaped the way they are.

Let’s begin by looking at a linear equation such as $y = 2x - 4$, the graph of which is shown in Figure 11.

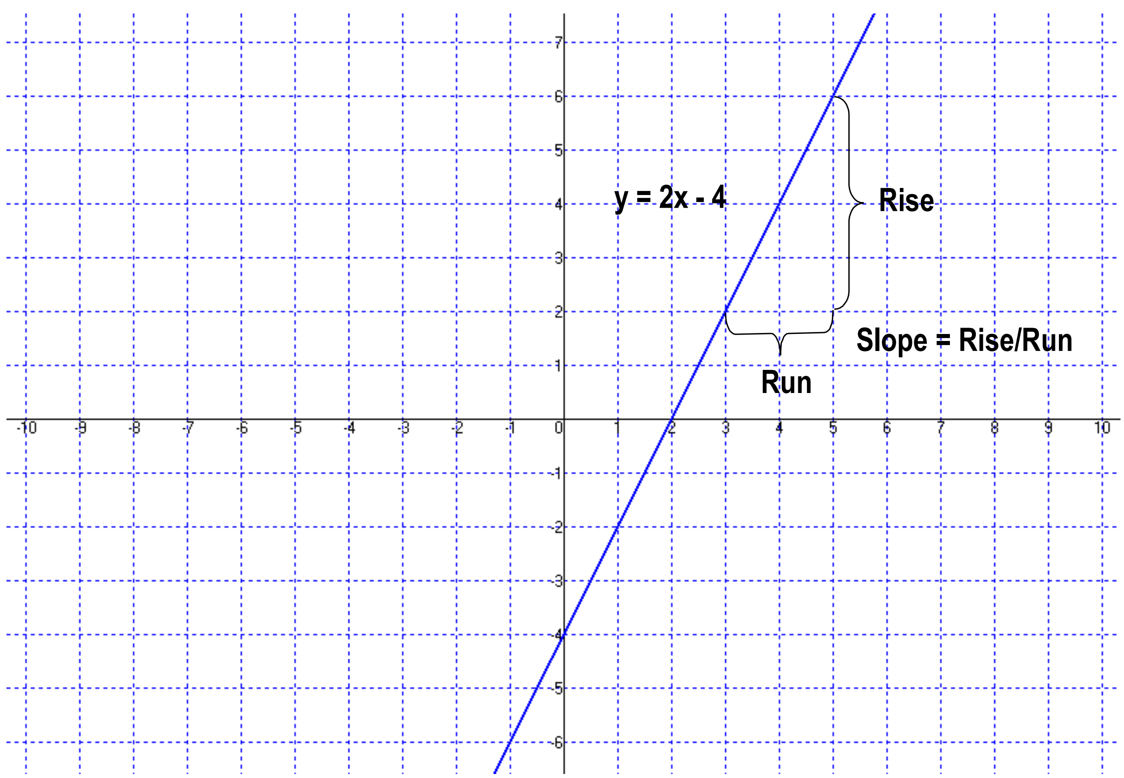

Figure 11: The graph of $y = 2x - 4$.

This type of equation is easy to graph. First, let x = 0, and substitute in the equation. This gives y = –4. Put a dot at (0, –4) (locations on a 2 dimensional graph are given in the form (x, y), that is, the x value and then the *y* value). Next substitute for y = 0 and solve the equation for *x*, which gives 2 for this equation. Put a dot at (2, 0). Now, draw a straight line between the two dots and you have your graph.

The part of this equation that we want to examine is what is called the *slope*. The slope is defined as the rise of the line over the equivalent run. Looking at the graph, you'll notice that we start at y = 2 (specifically, the point (3, 2)) and go over two units to the point (5, 2). Now we move upward until we encounter the line again, and count how many units we move. In this example, we move upward 4 units. So our run was 2 units and our rise was 4 units. Since slope is defined as rise over run, the slope is equal to 4/2 or 2.

So what does this tell us? We now know that no matter what point we choose on the graph (for any value of *x*) the graph will move upward on the *y* axis by two units if we move one unit to the right on the *x* axis. This may seem like a trivial point, and it is for a simple equation like y = 2x – 4. Things get much more complex, however, when the equation contains a value of *x* raised to a power other than 1. In fact, an equation which only contains x^1 (no powers of *x* other than 1 or 0) is known as a linear equation. Any equation with powers of *x* other than 1 (or 0) is known as a non-linear equation. While linear equations are easy to solve, they are only a small set of all possible equations.

What I'm going to explain next is how to find the "slope" of a non-linear equation and why knowing the "slope" allows you to solve certain kinds of problems. Let's begin by looking at the fairly simple equation $y = x^2 + 1$, the graph of which is shown in Figure 12.

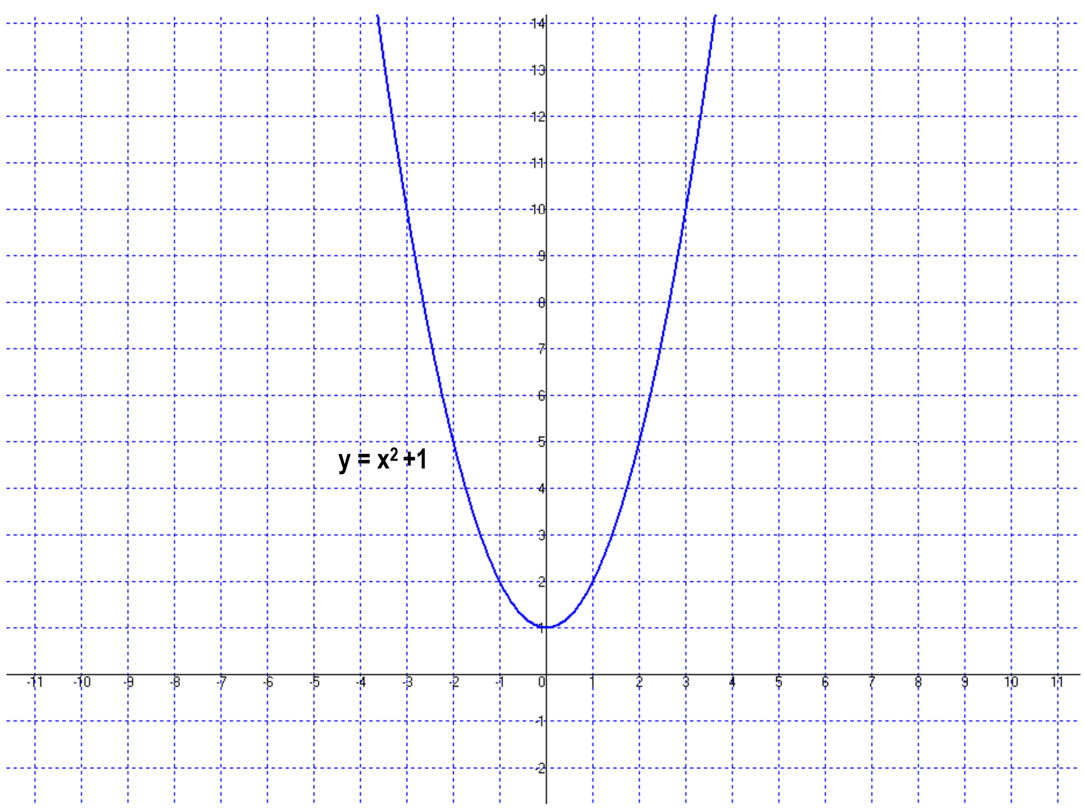

Figure 12: The graph of $y = x^2 + 1$.

The first thing we notice is that the graph is not a straight line – it's a curve. So defining the slope as the rise over the run will be a problem because if we choose a small run, the rise may be small. But if we choose a large run, the rise may be significantly much bigger. So let's define the slope at any point on the curve to be the slope of a tangent line drawn at that point. See Figure 13 where I've attempted to draw a tangent line to the point (1, 2). I just "eyeballed" this tangent line so we should not expect it to be exactly accurate.

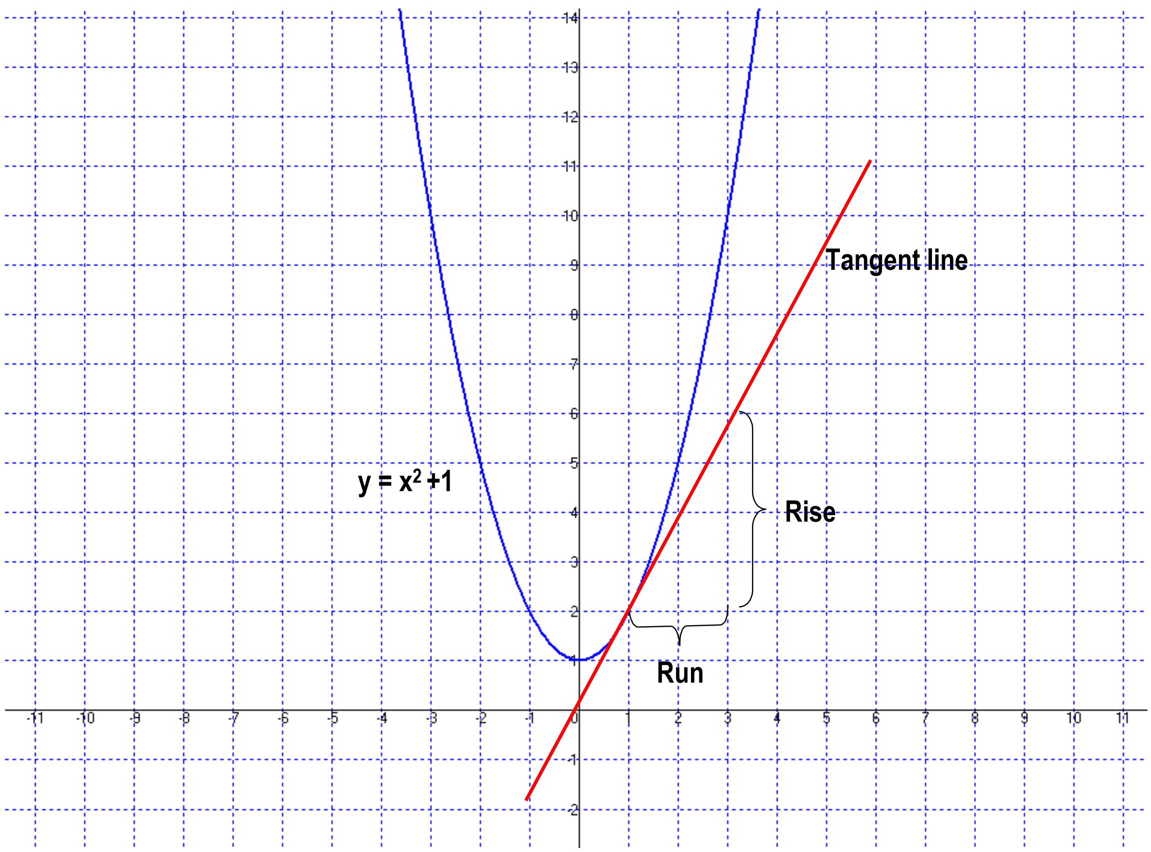

Figure 13: A tangent line drawn at the point x = 1, y = 2 of the graph of $y = x^2 + 1$.

The question differential calculus addresses is whether there is a general mathematical procedure for calculating the slope at a give point of a non-linear curve. We'll do this by using the concept of limits.

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The Differential of x^n

Let's start with a specific example, instead of the general case. The example we'll use is the equation of the graph in Figure 13.

$$y = x^2 + 1$$

Remember that we defined the slope to be the rise over the run. So we want to take a small segment of *x* and then calculate what the corresponding segment of *y* is. We can express this mathematically as

$\Delta y/\Delta x$ (Δ means "a small amount" and is read "delta".)

So, let's see what the equation will be if we extend *x* and *y* by Δx and Δy. To do this, we substitute into our equation, using $(x + \Delta x)$ for x and $(y + \Delta y)$ for *y*.

$$y + \Delta y = (x + \Delta x)^2 + 1$$

Now, expand the square.

$$y + \Delta y = x^2 + 2x\Delta x + \Delta x^2 + 1$$

This defines a point just a little bit away from the original point, which was defined by our original equation $y = x^2 + 1$. To get the distance between the two points, we subtract the original point (our original equation) from the equation of the point just a little bit away from our original point.

$$(y + \Delta y = x^2 + 2x\Delta x + \Delta x^2 + 1) - (y = x^2 + 1)$$

$$(y + \Delta y) - y = (x^2 + 2x\Delta x + \Delta x^2 + 1) - (x^2 + 1)$$

Let's simplify.

$$\Delta y = 2x\Delta x + \Delta x^2$$

Now we want to find the slope, which is defined as the rise over the run. The rise is the change in *y*, which is Δy, and the run is the change in *x*, which is Δx. So what we want is $\Delta y/\Delta x$. We can get this on the left side of the equation by dividing both sides of the equation by Δx.

$$\Delta y/\Delta x = 2x + \Delta x$$

We now ask ourselves, what will the slope be as we make Δx smaller and smaller. Since Δx goes to zero, the slope goes to $2x$. To indicate that we've already let Δx go towards zero, we use a slightly different notation.

$dy/dx = 2x$

The dy/dx notation means the differential of *y* with respect to *x*.

Looking back at the graph of the equation $y = x^2 + 1$ in Figure 13, we ask ourselves what is the slope at $x = 1$, the place where I drew the estimated tangent line? We know the slope is $2x$, so when we substitute 1 for *x*, we see that the slope *at this particular point* is 2. If we look at a different point on the graph, perhaps $x = -3$, what will the slope be? Substituting into our slope equation, $dy/dx = 2x$, we find that the slope at the point $x = -3$ is -6. And, sure enough, if we eyeball a tangent line at $x = -3$, we see that its angle is opposite in direction from our other tangent line – it has a negative slope.

Extending our analysis to a more general form, we find that the differential of x^n is

$dy/dx\ (x^n) = nx^{n-1}$

This is true for all values of *n*, including $n = 1$ and $n = 0$.

When doing differential calculus, you can't take the time to do the $\Delta y/\Delta x$ analysis that I did above for every equation you encounter. So students are taught the differentials for different functions – you memorize them just like you memorized the multiplication tables. There are also books which give the differential of different functions, in case you run into one you haven't seen before.

If you've followed all of the above explanation, you now understand differential calculus. ***That's all there is to the differential calculus!*** The rest is just mopping up, primarily using algebra. Your calculus professors will give you much more complex equations to differentiate. In general, your task will be to reduce the equations to some elementary form which you know how to differentiate.

The Tin Can Problem

A reasonable question is "What good is all this differential calculus?" "What can I do with it?" Let me pose a question which is fairly easy to solve with differential calculus.

"Why are cans shaped the way they are?" Most cans seem to have a certain relationship between the height and the diameter. Why is this?

The answer is that there is a certain combination of height and diameter which will provide the smallest surface area, meaning that it will take the minimum amount of metal to make the can. Let's solve this problem using differential calculus.

Let x = the radius (note: radius, not the diameter) and h = the height of the can. The volume of the can will be

$$V = \pi x^2 h$$

The surface area will be the surface area of the top, plus the surface area of the bottom, plus the surface area of the cylinder. Note that the bottom and top have the same surface area.

$$A = 2 * (\text{surface area of top}) + \text{surface area of cylinder}$$

$$A = 2 * \pi x^2 + 2\pi x h$$

Or

$$A = 2\pi x^2 + 2\pi x h$$

So now we have two equations (one for volume and one for area) and two unknowns, the radius, *x*, and the height, *h*. Let's solve the volume equation for the height and then substitute for *h* in the area equation.

$$V = \pi x^2 h$$

$$h = V/\pi x^2$$

Substitute this into the area equation.

$$A = 2\pi x^2 + 2\pi x V/\pi x^2$$

Simplify

$$A = 2\pi\, x^2 + 2V/x$$

Or

$$A = 2\pi\, x^2 + 2Vx^{-1}$$

First, let's look at the graph of this equation. *V* is a constant so let's just set it to 1 to draw the graph. See Figure 14.

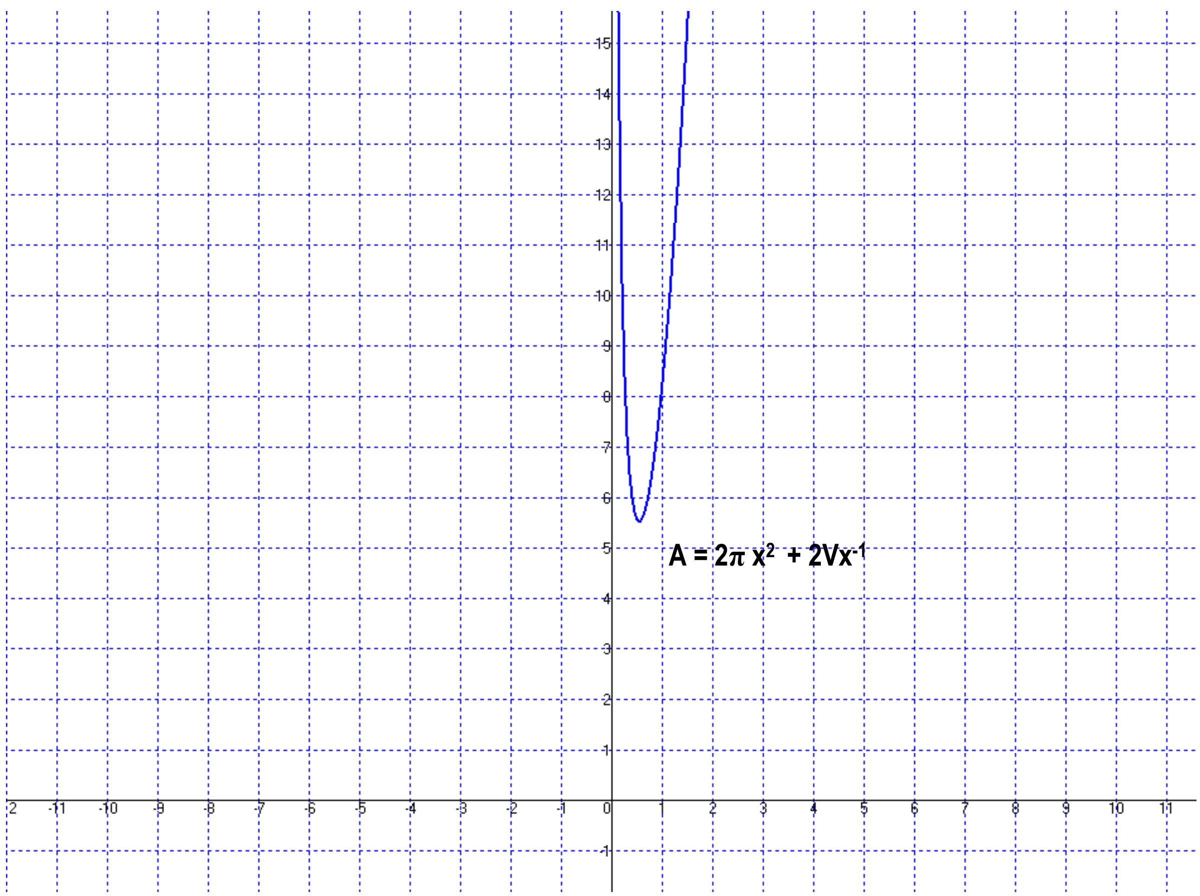

Figure 14: The graph of $A = 2\pi\, x^2 + 2Vx^{-1}$. Note the minima between x = 0 and x = 1.

Note that the area is extremely large for values of the radius close to zero. Since the height must go towards infinity to compensate for the radius going to zero, the area also goes towards infinity.

And as the radius gets larger than a certain value, the area gets very large because of the square term of the radius in the first term of the area equation. So there's a definite minimum point to the area. How can we find it?

Remember that by taking the differential, we are obtaining the slope of a tangent line to any point on the curve. If we look at that minima point on the curve and picture a tangent line at that exact point, what will it look like and what will its slope be? See Figure 15.

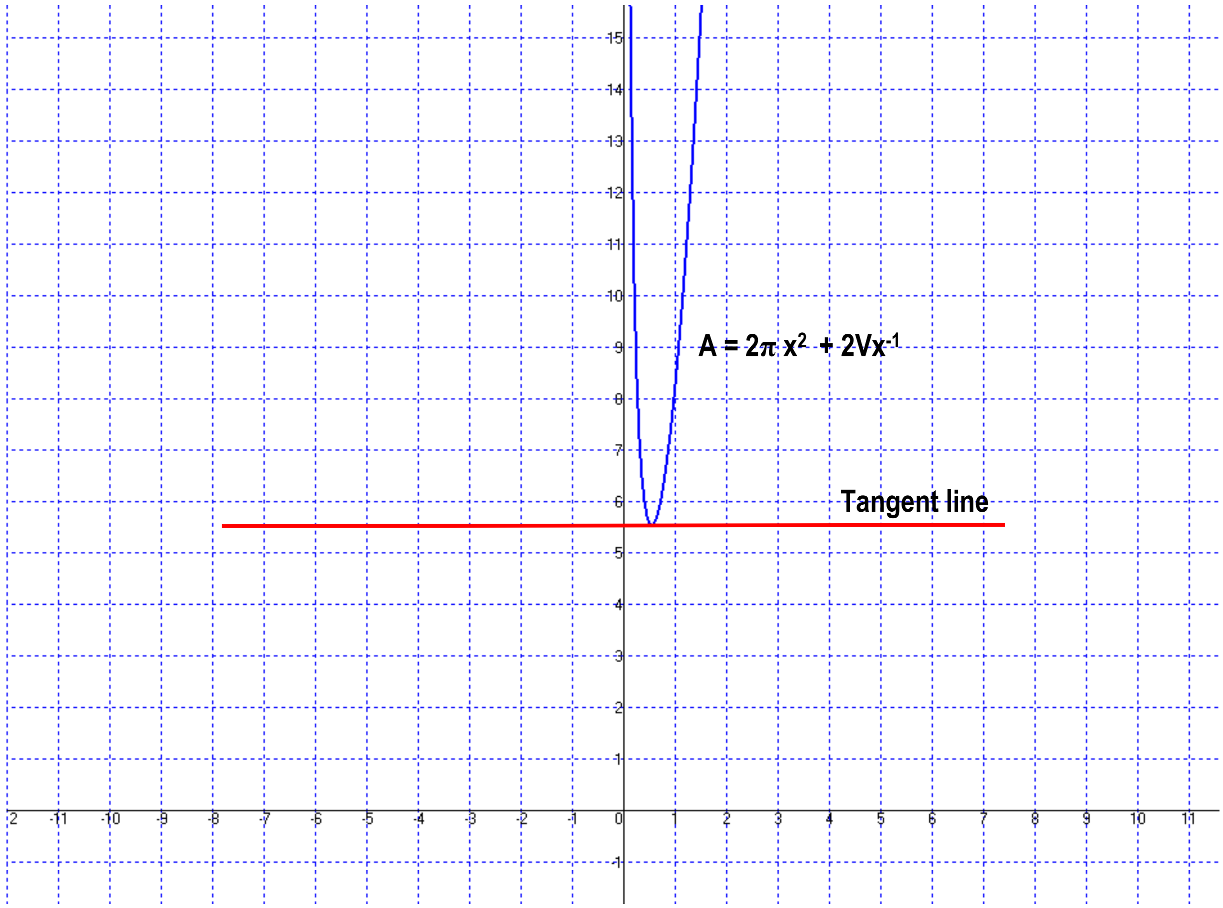

Figure 15: The graph of $A = 2\pi x^2 + 2Vx^{-1}$ with a tangent line drawn at the minima point. The slope of the tangent line is 0.

The slope is zero because there is no rise for any run. So to solve this equation, we take the differential, which gives us the equation for the slope at any point on the curve, and we solve that equation to find where the slope is zero.

So let's take the differential of our area equation. This is an equation with variables of the form x^n which we know how to differentiate.

$$A = 2\pi x^2 + 2Vx^{-1}$$

$$dA/dx = 2\pi (2x) + 2V(-1x^{-2})$$

$$dA/dx = 4\pi x - 2Vx^{-2}$$

dA/dx is the slope. Since we want to find the place where the slope is zero, we set dA/dx = 0 and solve the resulting equation.

$$0 = 4\pi x - 2Vx^{-2}$$

$4\pi x = 2Vx^{-2}$

$4\pi x = 2V/x^{2}$

$4\pi x^3 = 2V$

$x^3 = 2V/4\pi$

$x = (V/2\pi)^{1/3}$

Now that we know the radius, *x*, we substitute into the volume equation and solve for *h*.

$V = \pi x^2 h$

$V = \pi ((V/2\pi)^{1/3})^2 h$

After going through some algebraic simplification, this results in

$h = (4V/\pi)^{1/3}$

Now, let's look at the ratio of the radius to the height. Let's let *R* be the ratio.

$R = x/h$

$R = (V/2\pi)^{1/3}/(4V/\pi)^{1/3}$

$R^3 = (V/2\pi)/(4V/\pi)$

$R^3 = (V/2\pi)*(\pi/4V)$

Canceling like terms

$R^3 = 1/8$

or

$R = 1/2$

So the radius should be half the height of the can to minimize the volume, no matter what the volume. Or, stated another way, the diameter should be equal to the height of the can. Now, go to your pantry and see if cans are made that way. You'll find that most cans have a diameter that is less than the height. The reasons that many cans do not have an ideal proportion are primarily due to consumer preferences and tradition – the way cans have always been shaped.

The Differential of b^x

Before I leave differential calculus, there's one more derivative that I want to discuss. This is the derivative of a constant taken to the *x* power, or

$$y = b^x$$

Find dy/dx. As before, we examine a point Δx beyond *x*.

$$y + \Delta y = b^{x + \Delta x}$$

Now subtract the original equation.

$$y + \Delta y - y = b^{x + \Delta x} - b^x$$

$$\Delta y = b^x b^{\Delta x} - b^x$$

$$\Delta y = b^x (b^{\Delta x} - 1)$$

Now divide both sides by Δ*x*.

$$\frac{\Delta y}{\Delta x} = b^x \frac{\left(b^{\Delta x} - 1\right)}{\Delta x}$$

Or

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} b^x \frac{(b^{\Delta x} - 1)}{\Delta x}$$

Now, b^x has no Δx terms so we can remove it from the limit calculation.

$$\frac{dy}{dx} = b^x \lim_{\Delta x \to 0} \frac{(b^{\Delta x} - 1)}{\Delta x}$$

Now, here's a very important point. If $\lim_{\Delta x \to 0} \frac{(b^{\Delta x} - 1)}{\Delta x}$ exists (if the limit is not infinite), we can represent the limit by some constant, *k* ($k = \lim_{\Delta x \to 0} \frac{(b^{\Delta x} - 1)}{\Delta x}$). So the derivative of b^x is

$$\frac{dy}{dx} = kb^x$$

In other words, the derivative of a constant to the power *x* is equal to itself times a proportionality constant. If we can find a value for *b* that makes k = 1, we will have found a function whose derivative is equal to itself. To do that, we must find

$$\lim_{\Delta x \to 0} \frac{(b^{\Delta x} - 1)}{\Delta x} = 1$$

For simplicity, let's use *h* instead of Δx (so I don't have to keep writing Δx).

$$\lim_{h \to 0} \frac{(b^h - 1)}{h} = 1$$

One way for this specific limit equation to have a limit is if the equation equals 1 for finite values of *h*. Let's see if we can find that value.

$$\frac{(b^h - 1)}{h} = 1$$

$$b^h - 1 = h$$

$$b^h = h + 1$$

$$b = (h + 1)^{1/h}$$

Now, to get rid of the fractional 1/h power, let us use m = 1/h

$$b = (1/m + 1)^m$$

or

$$b = (1 + 1/m)^m$$

Now, since m = 1/h and we were letting $h \to 0$, we must now use $m \to \infty$. So we now ask, what is the limit of

$$b = \lim_{m \to \infty} (1 + \tfrac{1}{m})^m$$

But this equation is our compound interest equation, which has a limit of *e*. So to get k = 1, b = *e*. Thus

$$\frac{de^x}{dx} = e^x$$

In the more general case, where *x* is multiplied by some constant *a*, the derivative is

$$\frac{de^{ax}}{dx} = ae^{x}$$

This is another one of those really important equations in mathematics – it makes solving linear differential equations much simpler.

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Differential Equations

The first questions you may ask about differential equations are “Where do differential equations come from?” and “What kind of problems require the solution of differential equations?”

Differential equations are different from equations you’ve studied so far because the solution of a differential equation is a function (that is, an equation) rather than a number.

The Harmonic Oscillator

I’m going to analyze one system, that of a weight on the end of a spring as shown in Figure 16, known technically as a “harmonic oscillator.” Additionally, I’m going to start by making a simplifying assumption that there is no friction[10] in the system. While this is not a real world example, it will allow you to see how differential equations develop and how to solve for the equations of motion in this system. Later in this paper, we’ll add friction to the analysis and you’ll see how to solve slightly more complex systems.

[10] An alternate term for “friction” is “resistance”.

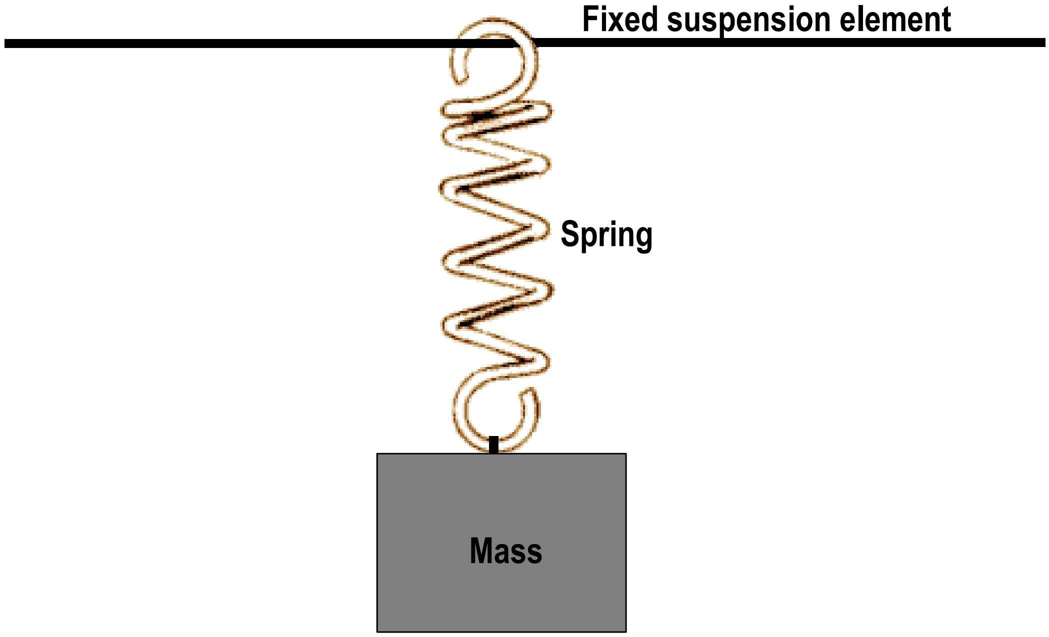

Figure 16: A system consisting of a weight on the end of a spring.

Why is this system important? It turns out that the equations we develop here can describe many different systems – a pendulum without friction, an electrical circuit containing only inductance and capacitance, and many other systems which oscillate. You'll run into this equation (and the one with resistance added) over and over again. So if you learn how to analyze a simple mechanical system, you'll be able to apply that analysis to many different systems.

Now, you probably already know a bit about what happens in this "weight on the end of a spring" system. If you pull it down and let it go, it will bounce up and down at a certain rate – the system has a natural frequency of oscillation. The reason it acts this way has to do with the inertia of the mass. When you let it go, after pulling it down, the mass is accelerated upward by the force of the spring. When the mass reaches the "rest position" however, it is moving at some velocity and continues pass the rest position due to inertia. Once past the rest position, the mass begins to compress the spring, meaning that the spring is now pushing against the mass, slowing it down. Eventually, the mass comes to rest but the spring has been compressed and is applying a force against the mass, causing it to begin moving downward.

What's actually happening is that energy is being transferred between the spring and the moving mass. When we pulled the weight down, we put energy into the spring. When we let go of the weight, the spring began to transfer energy to the mass in the form of kinetic energy (the motion of the mass). The movement of the mass deformed the spring, transferring energy back into the spring, etc. So the energy in the system transitions from kinetic energy to potential energy, and back again. And since we assumed no friction or other losses, the weight just keeps bobbing up and down forever.

But now, let's start from first principles and develop the equations for this system. We'll start our investigation with a weight dropped from some height. Let's assume that you have the measuring tools to accurately measure time and the speed of the weight. You find a high place where you can drop a weight and measure the time it takes to reach the bottom and the speed of the weight. Maybe you take a trip to Pisa, Italy and use the leaning tower of Pisa, or to Kuala Lumpur, Malaysia to the Petronas Twin Towers.

In any case, you perform experiments and find the following graph, Figure 17, of distance in feet versus time in seconds.

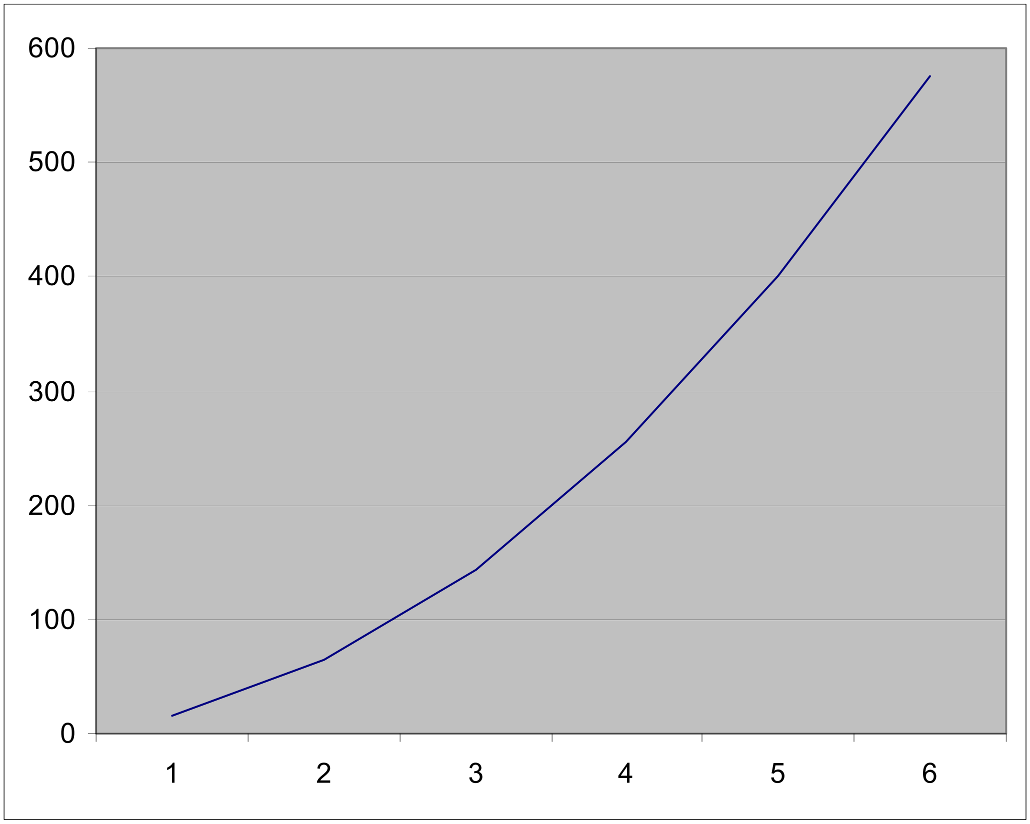

Figure 17: The graph of distance fallen in feet verses time in seconds.

From your data, we see that the weight fell 16 feet in the first second. Although not shown on the graph, you measured the speed of the falling weight after one second as 32 feet per second. After two seconds, the weight had fallen 64 feet and was traveling 64 feet per second. Using these numbers, you construct an equation for the distance, as

$$x = 16\, t^2 \quad \text{(feet)}$$

You find that this equation accurately predicts the distance fallen by the weight for each of your measurement points. Then you look at your speed (or velocity) values and by cutting and trying, you construct an equation for the velocity as

$$v = 32\, t \quad \text{(feet per second)}$$

And since you found that the speed was 32 feet per second after one second, you know that the acceleration was

$$a = 32 \quad \text{(feet per second per second)}$$

Thinking back on your differential calculus studies, you notice that the velocity is the differential of the distance and the acceleration is the differential of the velocity.

$$v = \frac{dx}{dt}$$

$$a = \frac{dv}{dt}$$

Substituting for *v* in the above equation,

$$a = \frac{d(\frac{dx}{dt})}{dt}$$

The last equation can also be written as

$$a = \frac{d^2 x}{dt^2}$$

which indicates the "second differential" of *x* with respect to *t*. You take the first differential, and then take the differential of that result.

And when you think about it, it makes sense that the velocity would be equal to the rate of change of the distance. If the slope of the distance curve were constant (if the distance curve were a straight line), the velocity would be constant. But if the slope of the distance curve is not constant, it means that the velocity is changing over time.

It may be easier to visualize this for velocity. If the velocity changes over time, it means there was an acceleration (either positive or negative) – it's the only way the velocity can change. Remember Newton's first law of motion, "A body in a state of motion will remain in that state of motion unless acted upon by an outside force."

A constant velocity would be represented by a horizontal line on a graph of velocity verses time. See the red line in Figure 18. Since a horizontal line has a zero slope, the line indicates that there is no acceleration. But if the velocity line has a slope on the graph, as indicated by the blue line in Figure 18, there must be an acceleration and the equation of the velocity will have a non-zero derivative. Look at the equations of the lines indicated next to the red and blue lines. Note that the derivative of the equation for the red line is zero, while the derivative for the equation for the blue line is 0.5.

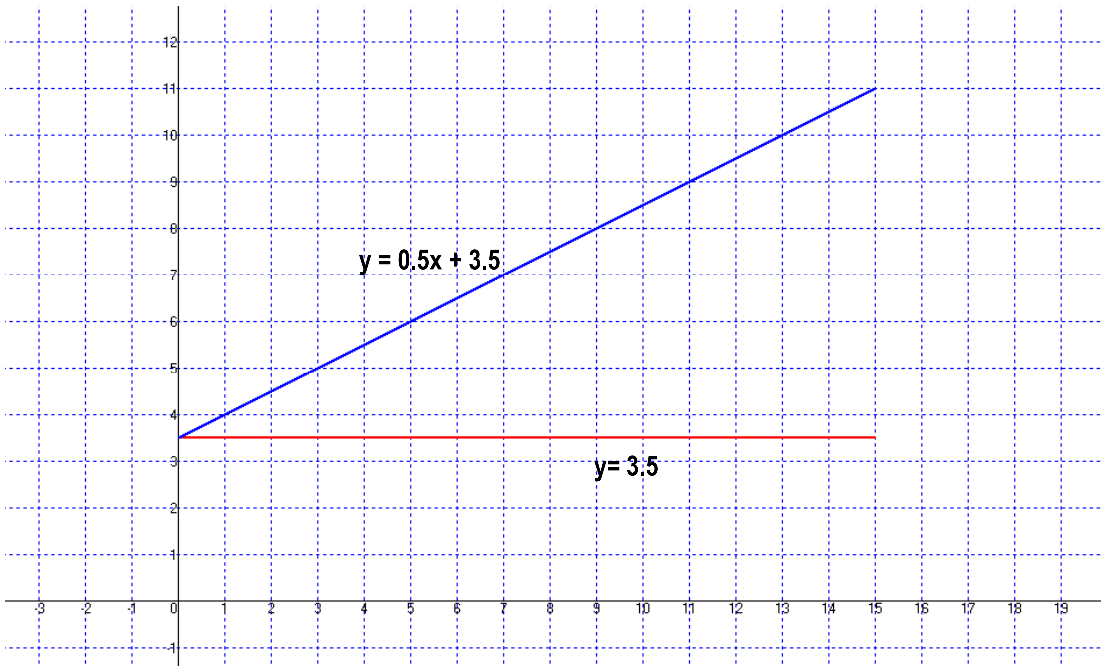

Figure 18: Graph of velocity verses time. The red line indicates a constant velocity (no acceleration) while the blue line indicates an increasing velocity, meaning that acceleration is occurring.

Newton's second law of motion will also be important to our analysis: "The relationship between an object's mass m, its acceleration a, and the applied force F is $F = ma$."[11]

Now, in the analysis we did above, we found several different ways to represent the acceleration.

$$a = \frac{dv}{dt}$$

and

$$a = \frac{d^2x}{dt^2}$$

If we substitute into Newton's second law of motion

$$F = ma$$

we get

$$F = m\frac{dv}{dt}$$

and

$$F = m\frac{d^2x}{dt^2}$$

[11] And just for completeness, Newton's third law of motion is, "For every action there is an equal and opposite reaction."

These are differential equations. Now let's see how we can use Newton's laws of motion and these differential equations to analyze our "weight on the end of a spring" system of Figure 16 and Figure 19.

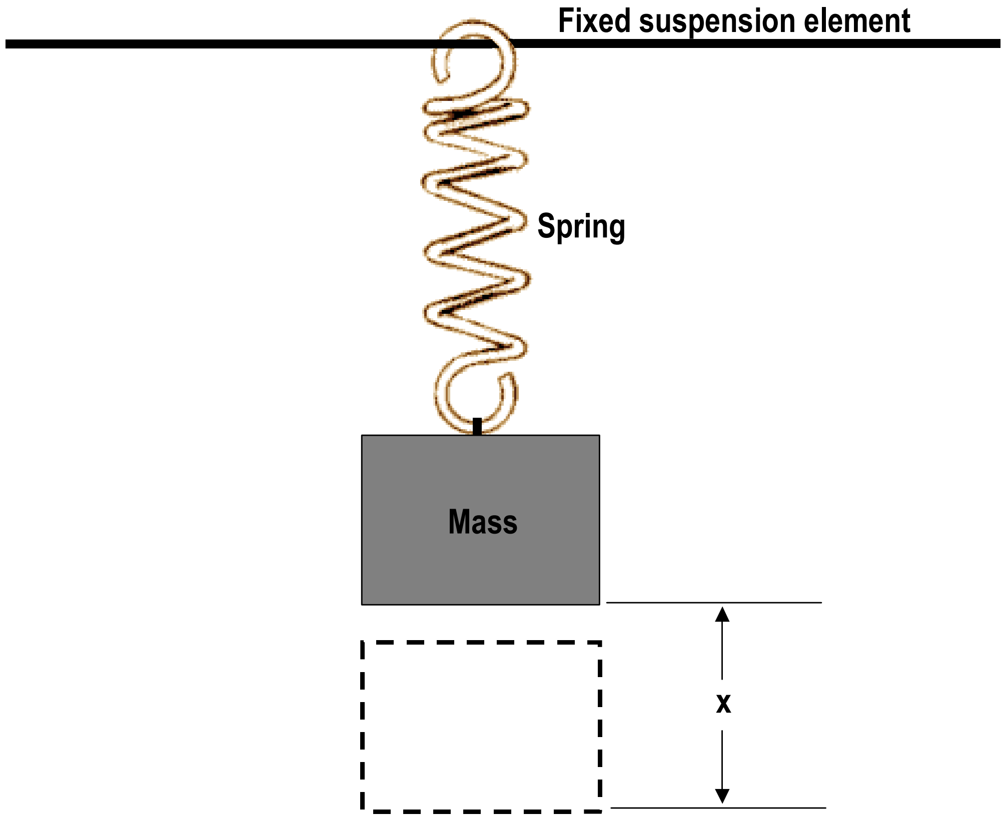

Figure 19: The "weight on the end of a spring" system, with the weight pulled down. When the weight is release, the system will begin oscillating.

The first thing we recognize is that, by Hooke's law[12], the force of the spring is proportional to the amount of deflection. That is, the spring force is equal to *kx*. And since we pulled it down, we'll use a negative sign to indicate that the force is pulling upwards. Substituting into Newton's second law equation, $F = ma$, we have

$$-kx = m\frac{d^2x}{dt^2}$$

or

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

[12] Named for Robert Hooke, English mathematician and scientist (1635 – 1703).

So how do we solve this equation? We start by assuming that x = $e^{i\omega t}$. Remember from our analysis earlier that the real part of $e^{i\omega}$ is equal to cos ωt. So we're assuming that the motion of the weight will be equal to a cosine function, if we take the full swing as equal to 2 units (because the cosine goes from +1 to –1).

Now, let's substitute $e^{i\omega t}$ for x in the equation above.

$$\frac{d^2(e^{i\omega t})}{dt^2} = -\frac{k}{m}e^{i\omega t}$$

Let's get to the second differential. Initially, we'll take the first differential.

$$\frac{d(e^{i\omega t})}{dt} = i\omega e^{i\omega t}$$

To get to the second differential, we take the differential of the first differential.

$$\frac{d(i\omega e^{i\omega t})}{dt} = (i\omega)^2 e^{i\omega t}$$

or

$$\frac{d^2(e^{i\omega t})}{dt^2} = -\omega^2 e^{i\omega t}$$

So our original equation

$$\frac{d^2(e^{i\omega t})}{dt^2} = -\frac{k}{m}e^{i\omega t}$$

becomes

$$-\omega^2 e^{i\omega t} = -\frac{k}{m}e^{i\omega t}$$

The first thing we see is that we can divide both sides by $e^{i\omega t}$ leaving

$$-\omega^2 = -\frac{k}{m}$$

Or, since we have a negative sign on both sides,

$$\omega^2 = \frac{k}{m}$$

We're going to have more *ω's* later so we'll designate this one by a zero subscript[13].

[13] ω_0 is the resonate frequency of the system, in radians per second.

$$\omega_0^2 = \frac{k}{m}$$

Remember that we defined $\omega = 2\pi f$, so

$$2\pi f = \sqrt{\frac{k}{m}} \text{ radians}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \text{ hertz}$$

Note that the oscillation frequency is totally determined by the spring constant and the weight. It is not determined in any way by how far you pull the weight down before you let it go. A "weight on the end of a spring" system has a natural frequency of oscillation that is completely determined by just two factors, k and m.

If you increase the mass of the weight, the system will oscillate slower. And if you make the spring stronger, the system will oscillate faster.

So the solution to this equation is $e^{it\sqrt{\frac{k}{m}}}$ or $\cos t\sqrt{\frac{k}{m}}$ (since we only look at the real part in the real world).

Congratulations! You just solved your first differential equation!

Now, some bad news. The solution given above is not the only solution to the equation. Suppose we had used a constant times $e^{i\omega t}$, such as $Ae^{i\omega t}$. If you do the above math again with $Ae^{i\omega t}$ instead of $e^{i\omega t}$ you'll find that the solution is just as valid (see Appendix B).

It turns out that A is the initial displacement of the weight and determines the amplitude of the swing. Its value depends upon what system of units we're using but suppose we're using the English system and we pulled the weight down 2 inches from it's resting point. A would be 2 and the weight would swing four inches, from +2 inches to –2 inches, with the resting point being the zero location.

Now, even this solution is not the most general solution to the equation. Suppose we initialized the system by giving the weight an impulse while it was in the rest position. The motion would not be a cosine, but would be a sine function since it starts at the zero position instead of someplace below the zero position. However, the sine function is equivalent to the cosine plus π so we can represent this solution by

$$x = Ae^{i(\omega+\pi)t}$$

There are even more possible solutions, which all depend upon differing initial conditions. I won't go into these other solutions here but if you study physics in college, you may encounter them in your classes. For now, celebrate that you've learned how to solve a (simple) differential equation. For a high school student, that's a significant accomplishment – congratulations!

The Forced Harmonic Oscillator

If your objective is to learn the basics of differential equations, you can stop here. The sections which follow go into more detail on the harmonic oscillator and the mathematics gets a bit more complex. If you followed the mathematics to this point, you'll have no difficulty understanding differential equations in your college courses. The professors will lead you into the subject and you'll be able to extrapolate from what you know already to the more complex systems. But for those of you who want to explore further, let's roll!

Now we're going to look at some more complex, but more realistic systems. We're going to start with a system with a forcing function and then introduce friction, or resistance, into the system. And then we'll close by examining the transient behavior of a system with resistance.

So what do we mean by a system with a "forcing function?" A good example is a playground swing. The swing is a pendulum with a natural frequency. If the person on the swing moves just right, in time with the natural movement of the swing, the arc of the swing will increase. But if the person doesn't move just right, for example if they move at random, the arc will not increase and may even decrease. The action of the person on the swing is the forcing function.

In our system, the forcing function will be movement of the suspension element up and down with a fixed period of swing.

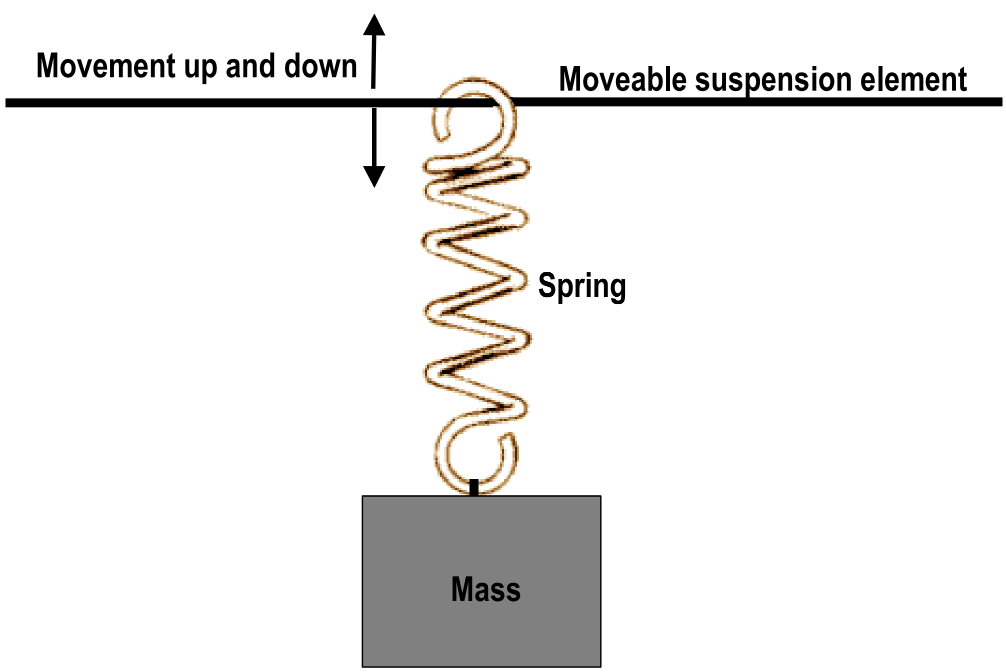

Figure 20: The "weight on the end of a spring" system, with the suspension element moving up and down at a constant rate.

So what will happen to the system shown in Figure 20, if we assume that the suspension bar is moved up and down in some regular fashion?

First, we want to make an assumption about the up and down movement of the suspension element (the forcing function). Let's assume that it's a regular sinusoid, perhaps a cosine function. We can represent such a function by the equation:

Forcing function = F $e^{i\omega t}$

F is the initial value of the force. In the general case, F can be a complex value, but let's ignore that for now. We're also not going to examine what happens when the forcing function is first applied – there will be a transient response as the weight begins to move. We're only going to examine what the response will be after a relatively long time after the application of the forcing function, i.e., the steady state response of the system.

Remember that we used Newton's second law of motion to analyze the original system.

$$F = ma$$

We still use that same equation, but the force is now the force of the spring plus the force of the forcing function.

$$Fe^{i\omega t} - kx = m\frac{d^2x}{dt^2}$$

Let's rearrange this to put it in a more common form.

$$m\frac{d^2x}{dt^2} = -kx + Fe^{i\omega t}$$

Divide through by the mass.

$$\frac{d^2x}{dt^2} = -\frac{kx}{m} + \frac{F}{m}e^{i\omega t}$$

$$\frac{d^2x}{dt^2} + \frac{kx}{m} = \frac{F}{m}e^{i\omega t}$$

In a manner similar to the way we solved our original differential equation, we assume a solution of x = $De^{i\omega t}$ and substitute in to the equation. Here, *D* represents the maximum magnitude of the movement of the mass, from the rest position (think of D meaning "displacement"). That is, if the mass moves a total of four inches, peak-to-peak, *D* will equal two. In reality, D will be a complex number because the mass will not move in phase with the forcing function. The forcing function will move the suspension element which will change the spring force by making the spring longer or shorter. This changed spring force will be applied to the mass, which will change its motion in response to the change in force. But the mass has inertia and it will not change its motion instantaneously. Thus, *D* must accommodate this phase shift, which it can do be being a complex number with a phase angle. More on this later.

$$\frac{d^2(De^{i\omega t})}{dt^2} + \frac{k}{m}De^{i\omega t} = \frac{F}{m}e^{i\omega t}$$

$$-\omega^2 De^{i\omega t} + \frac{k}{m} De^{i\omega t} = \frac{F}{m} e^{i\omega t}$$

Now we can divide by $e^{i\omega t}$.

$$-\omega^2 D + \frac{k}{m} D = \frac{F}{m}$$

Factoring

$$D(\frac{k}{m} - \omega^2) = \frac{F}{m}$$

$$D = \frac{F/m}{k/m - \omega^2}$$

Note, however, that k/m = ω_0^2 from our solution of the original system. Since neither the spring constant nor the mass changed for this equation, the natural resonant frequency is still the same.

$$D = \frac{F/m}{\omega_0^2 - \omega^2}$$

$$D = \frac{F}{m(\omega_0^2 - \omega^2)}$$

So the amount of movement of the mass is related to how close the frequency of the forcing function is to the natural resonant frequency of the spring and mass system. As the frequency of the forcing function approaches the natural resonant frequency, the displacement of the mass grows larger and larger, going to infinity.

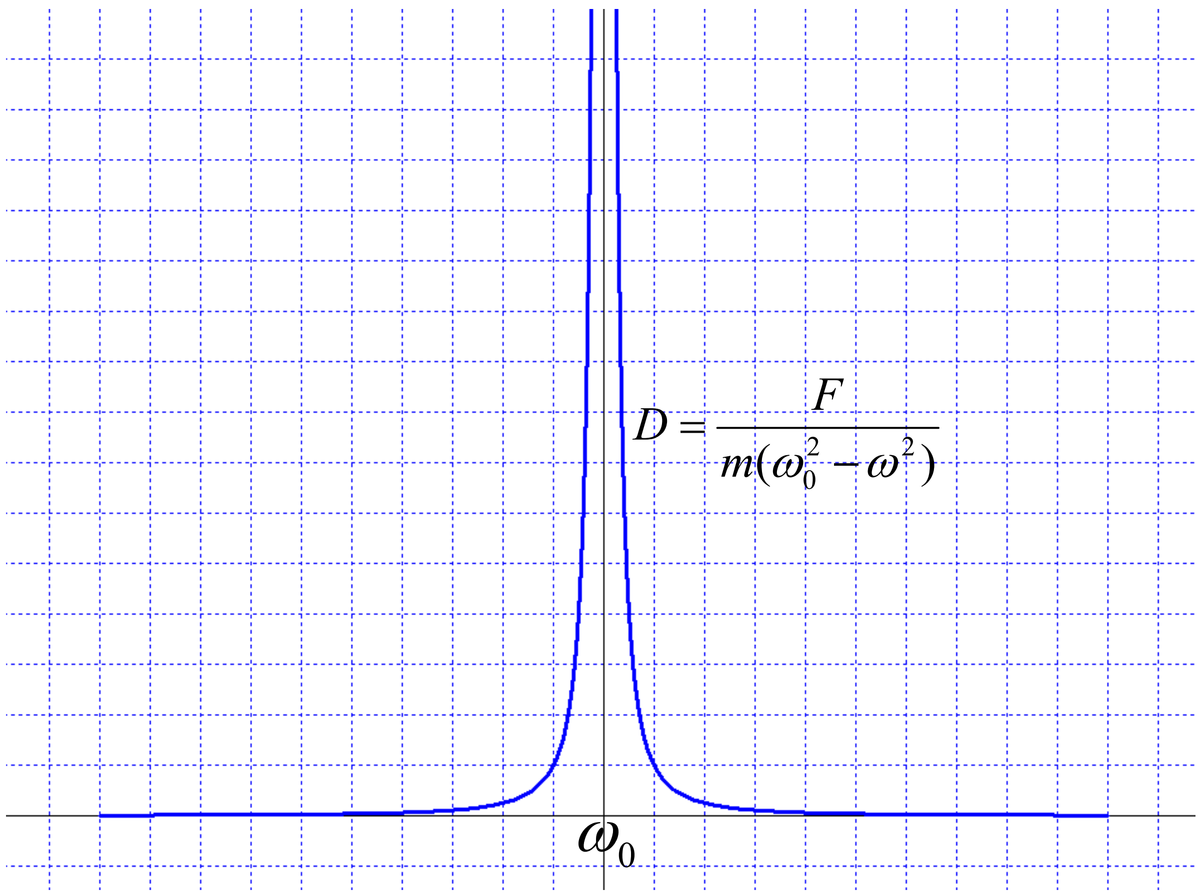

Figure 21: The displacement of the mass of Figure 20 as the frequency of the forcing function approaches the natural resonant frequency of the system.

This effect if exploited in electrical circuits to tune a circuit to a specific frequency, for example in a radio. It turns out that these same equations describe electrical circuits. Toward the end of this paper I'll discuss how to convert these equations to describe electrical circuits.

The Forced Harmonic Oscillator with Friction

But there's something wrong with this picture. In the real world, we don't see systems achieve infinite displacement when they oscillate. The reason is that we left out resistance in our analysis. Let's now add resistance to our analysis and see what results.

We're going to continue to use the concept of the harmonic oscillator with forced oscillation (with a forcing function). The reason we're going to use the harmonic oscillator with forced oscillation is that without the forced oscillation the movements of the mass would gradually die away as energy is converted to heat. What we'd then be studying is the transient response of the system, rather than the steady state response. Later, we'll analyze the transient response, but now, we're going to examine the addition of resistance in the simplest system possible.

For many mechanical systems, resistance (or damping), is a function of the speed (or velocity) of the motion. Since velocity is the first derivative of the distance, we can write the damping term as d = cv

Where c is a damping constant. Now, let's write the velocity as the derivative.

$$d = c\frac{dx}{dt}$$

Introducing this damping factor into our equation, we have

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = Fe^{i\omega t}$$

For algebraic simplicity the constant c can be replaced by γ times the mass. And since ω_0^2 = k/m , the spring proportionality factor k can be replaced by k = mω_0^2

$$m\frac{d^2x}{dt^2} + \gamma m\frac{dx}{dt} + m\omega_0^2 x = Fe^{i\omega t}$$

Factoring out the mass, we wind up with

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \frac{F}{m}e^{i\omega t}$$

Now, if we again assume the solution of x = $De^{i\omega t}$ and substitute into the equation.

$$\frac{d^2(De^{i\omega t})}{dt^2} + \gamma\frac{d(De^{i\omega t})}{dt} + \omega_0^2(De^{i\omega t}) = \frac{F}{m}e^{i\omega t}$$

$$-\omega^2 De^{i\omega t} + \gamma(i\omega)De^{i\omega t} + \omega_0^2 De^{i\omega t} = \frac{F}{m}e^{i\omega t}$$

Divide by $e^{i\omega t}$

$$-\omega^2 D + \gamma(i\omega)D + \omega_0^2 D = \frac{F}{m}$$

Now, factor out D.

$$D(-\omega^2 + \gamma(i\omega) + \omega_0^2) = \frac{F}{m}$$

$$D = \frac{F}{m(\omega_0^2 - \omega^2 + i\gamma\omega)}$$

So the magnitude of oscillation doesn't grow to infinity – it's limited by the imaginary term $i\gamma\omega$. Even when the frequency of the forcing function is exactly equal to the natural frequency of the harmonic oscillator, the denominator does not go to zero but has the imaginary value $i\gamma\omega m$.

But how are we to interpret D? Let's start by normalizing D, which we can do by dividing both sides by F. What we really want to do is analyze the term

$$\bar{D} = \frac{D}{F} = \frac{1}{m(\omega_0^2 - \omega^2 + i\gamma\omega)}$$

Where we use the notation $\bar{D}$ to indicate normalization. The first thing we should recognize is that $\bar{D}$ is a complex number. Let's analyze it in polar notation. We can obtain the magnitude, or modulus, of a complex number by multiplying the number by its conjugate.

$$r^2 = \frac{1}{(m(\omega_0^2 - \omega^2) + i\gamma\omega m)(m(\omega_0^2 - \omega^2) - i\gamma\omega m)}$$

$$r^2 = \frac{1}{m^2(\omega_0^2 - \omega^2)^2 - i^2\gamma^2\omega^2 m^2}$$

$$r^2 = \frac{1}{m^2[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2]}$$

Let's plot r^2, the magnitude of this complex factor. See Figure 22.

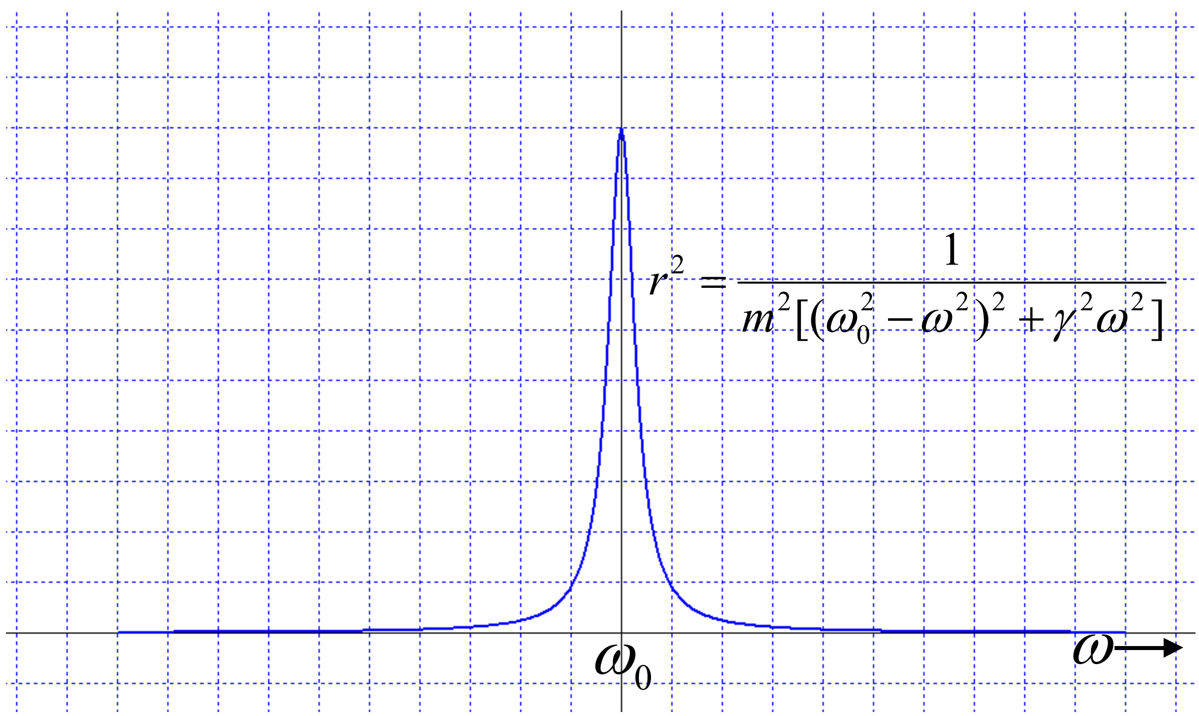

Figure 22: Plot of r^2, the magnitude of the complex factor, verses ω.

Now, we ask about the phase between the forcing function and the movement of the mass. Intuitively, we know that if the frequency of the forcing function is very much less than the resonant frequency, the mass will essentially move in phase with the forcing function. In such a case, the spring acts almost like a solid connector between the suspension element and the mass. As I slowly pull the suspension element upwards, for example, the mass follows along with very little delay compared to the time of one cycle of the forcing function. So when $\omega \ll \omega_0$, the phase difference is slightly negative, but very close to zero.

So now, let's look at the complex number that is $\bar{D}$ a bit closer.

$$\bar{D} = \frac{1}{m(\omega_0^2 - \omega^2 + i\gamma\omega)}$$

A complex number can be represented by the equation.

$$z = re^{i\theta} = r\cos\theta + i\sin\theta$$

And if you remember from our previous discussions of complex numbers, this will be represented by a vector in the complex plane that is rotated counterclockwise from the positive real axis. Suppose we take the reciprocal of a complex number, 1/z, how will this be represented in the complex plane? Let's work it out.

$$\frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{e^{-i\theta}}{r} = \frac{1}{r}\cos\theta - i\sin\theta$$

So the difference in phase between a complex number and its reciprocal is that the radius vector rotates in the opposite direction. Note that although the magnitude of the radius vector changes when we take the reciprocal (from r to $1/r$), the absolute value of the phase angle does not (although it is in the opposite direction). So we have a short cut for computing the phase angle of the reciprocal of a complex number – just change the sign of the imaginary term (the b term in $z = a + ib$) and calculate the phase angle as before.

$\bar{D}$ is the reciprocal of a complex number.

$$\bar{D} = \frac{1}{z} = \frac{1}{a+ib} = \frac{1}{m(\omega_0^2 - \omega^2) + i\gamma\omega m}$$

So to calculate the phase angle, we change the sign of the b term and calculate the phase as:

$$\tan\theta = \frac{-b}{a} = \frac{-\gamma\omega m}{m(\omega_0^2 - \omega^2)} = \frac{-\gamma\omega}{(\omega_0^2 - \omega^2)}$$ Note that we can cancel the mass, m.

Now, when $\omega < \omega_0$ the radius vector will be in the fourth quadrant, because b, the imaginary term, is negative and a, the real term, is positive. When $\omega = \omega_0$, the radius vector will be pointing directly down, with the vector lying on the negative imaginary axis, because the real term is zero. And with $\omega > \omega_0$, both the a and b terms will be negative, putting the radius vector in the third quadrant.

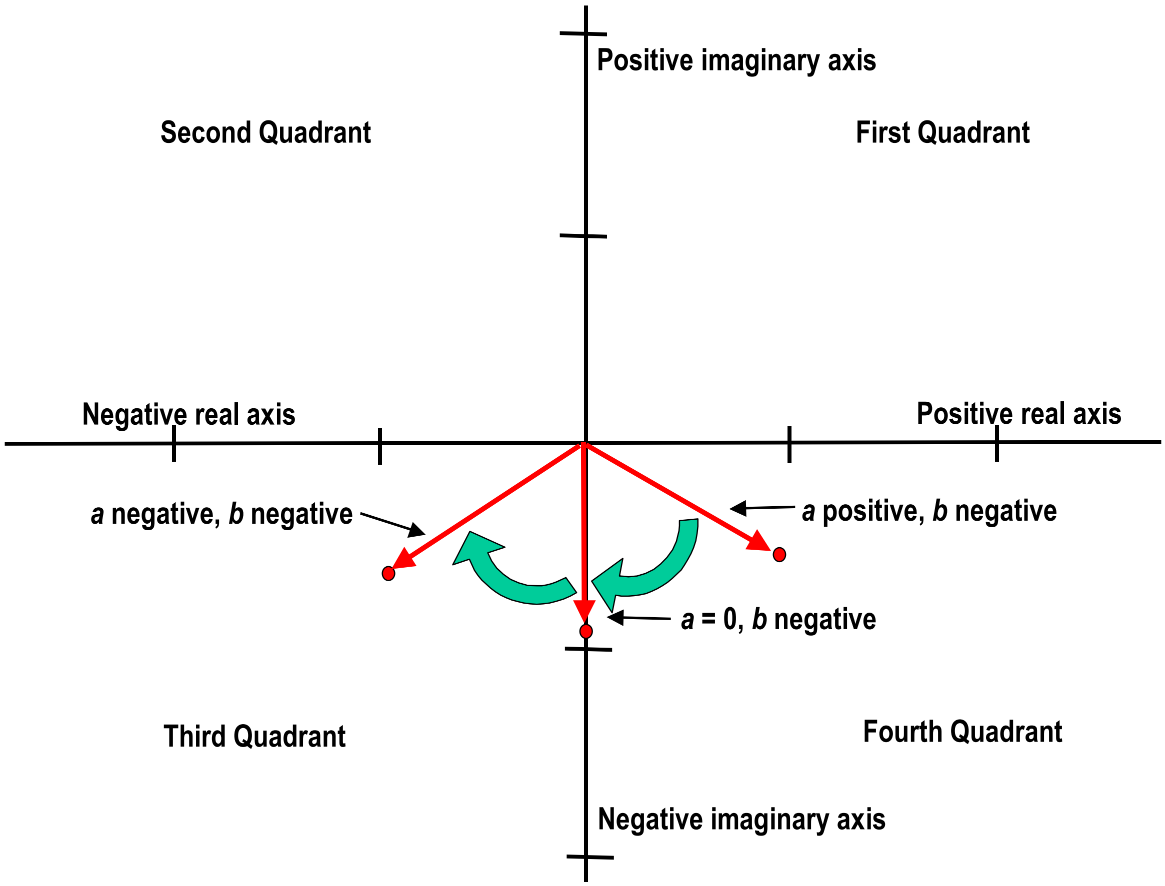

Figure 23: Plot showing how the phase changes as ω varies in relation to ω_0.

So we can see from the above figure, as ω increases, the radius vector will rotate clockwise, and the phase angle will change from approximately zero, to – 90 degrees, to almost –180 degrees. This is better shown in Figure 24.

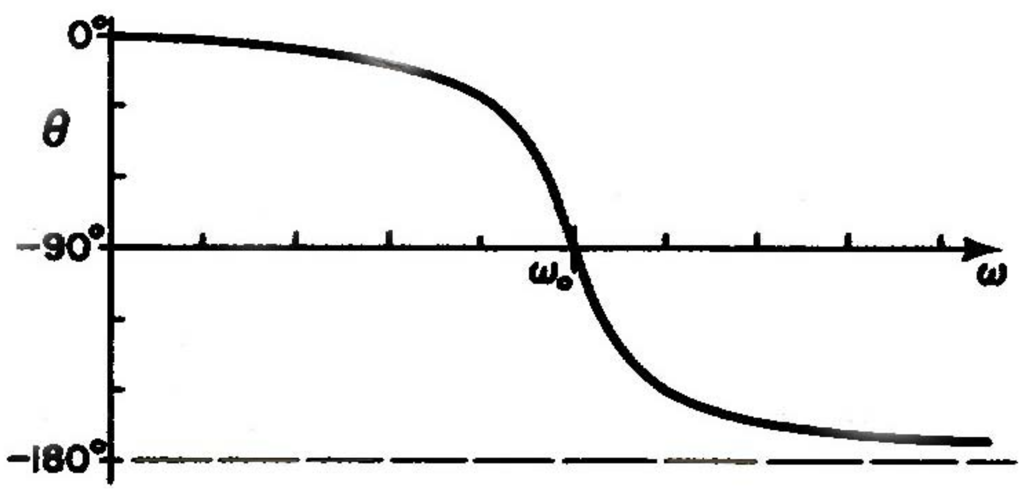

Figure 24: Plot of θ verses ω as ω varies in relation to ω_0.

What we see from this plot is that when the frequency of the forcing function is much less than the resonant frequency, the mass moves almost in phase with the forcing function. When the frequency of the forcing function is equal to the resonant frequency, the movement of the mass is 90 degrees behind the phase of the forcing function. And when the frequency of the forcing function is much greater than the resonant frequency, the mass is almost completely out of phase with the phase of the forcing function.

Next, I'm going to talk about the Q of the system. In order to do so, I'm going to assume that ω is very close to ω_0 and γ is very small. Let's look at our equation for $\bar{D}$ and see what these assumptions do for us.

$$\bar{D} = \frac{1}{m(\omega_0^2 - \omega^2 + i\gamma\omega)}$$

First we see that $(\omega_0^2 - \omega^2)$ can be factored as $(\omega_0 + \omega)(\omega_0 - \omega)$ and we can simplify this to $2\omega_0(\omega_0 - \omega)$.

$$\bar{D} \approx \frac{1}{2m\omega_0(\omega_0 - \omega + i\gamma/2)}$$

and

$$r^2 \approx \frac{1}{4m^2\omega_0^2[(\omega_0 - \omega)^2 + \gamma^2/4]}$$

If we plot this function verses ω we obtain a plot as shown in Figure 25.

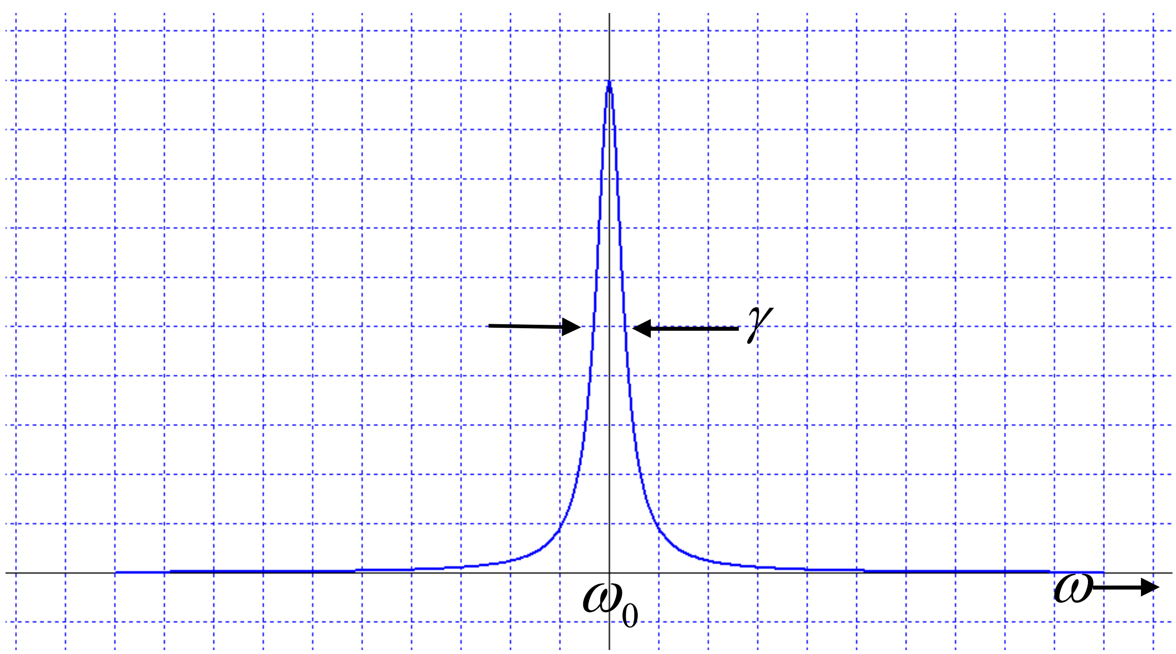

Figure 25: Plot of r^2 as ω varies in relation to ω_0.

It turn out that we can show that the width of the resonance curve, half way to the peak, is equal to γ, if γ is very small[14].

Based on this, Q is defined as

$$Q = \frac{\omega_0}{\gamma}$$

Thus, the resonance curve is taller and sharper as the frictional effects get smaller and smaller.

The Transient Response of the Harmonic Oscillator

At the beginning of this section on differential equations, I talked about how the harmonic oscillator works. Potential energy is put into the spring by pulling it down, and when you let go of the weight, the potential energy is transferred to the movement of the mass, which has kinetic energy. When the mass reaches the rest point, all of the energy has been transferred from the spring (the potential energy) to the movement of the mass (kinetic energy). However, the mass has inertia and it continues moving past the rest point, compressing the spring and transferring kinetic energy to the spring (potential energy). So in the basic harmonic oscillator without resistance, the mass just keeps bouncing up and down, with energy transferring between the compression of the spring (potential energy) and the motion of the mass (kinetic energy). See Figure 26.

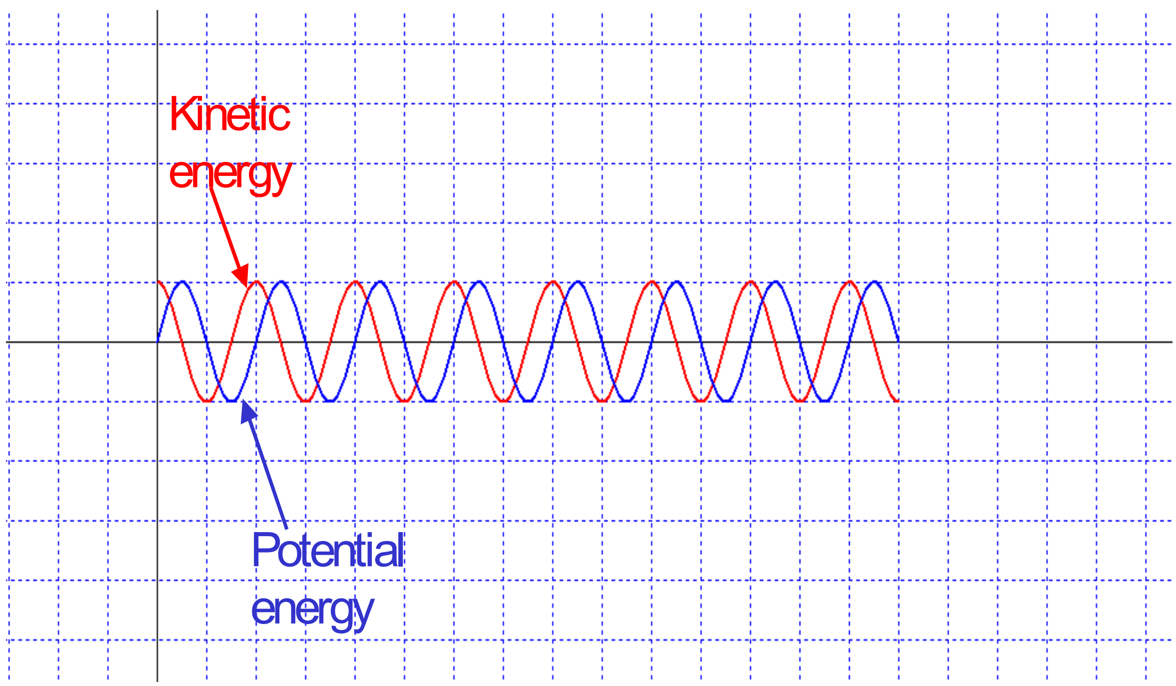

Figure 26: Potential and kinetic energy in the harmonic oscillator.

It's not immediately obvious from the figure, but the energy of the simple harmonic oscillator is constant. This is what we should expect from the principle of conservation of energy. The simple harmonic oscillator has no resistance so the energy put into the system has nowhere to go and thus remains

[14] See Appendix B for the mathematics which demonstrates this.

constant. The energy flows back and forth between potential energy and kinetic energy but maintains a constant value.

Now, I'm going to give you the equations for the potential and kinetic energy in a harmonic oscillator. The derivations require a level of calculus a bit beyond what I'm assuming for this paper. If you're interested in the derivations, they're available in any good physics book. But for now, just take them on faith.

$U = \frac{1}{2}kx^2$ Potential energy – k is the spring constant and x is the deflection.

$K = \frac{1}{2}mv^2$ Kinetic energy – m is the mass and *v* is the velocity.

In this section, we're going to examine what happens to the forced harmonic oscillator with friction, when we remove the forcing function. Since the system has friction, we can intuitively guess that the system will "run down" and eventually stop oscillating, as the energy of the system is dissipated in the resistance. What we're going to do is work out the mathematics for this "running down".

From our previous analysis, we know that the equation for the forced oscillator is

$$m\frac{d^2x}{dt^2} + \gamma m\frac{dx}{dt} + m\omega_0^2 x = Fe^{i\omega t}$$

The work done by the force, F, is the force times the velocity, and we know from our earlier analysis that we can write the velocity as dx/dt. So the work, or power, is

$$\text{Work} = F\frac{dx}{dt}$$

So our equation for the work of the forced oscillator is

$$Work = Fe^{i\omega t}\frac{dx}{dt} = m\frac{dx}{dt}\frac{d^2x}{dt^2} + m\omega_0^2 x\frac{dx}{dt} + \gamma m\left(\frac{dx}{dt}\right)^2$$

Now, because we want to express this in terms of kinetic and potential energy, we're going to express the equation as

$$Work = Fe^{i\omega t}\frac{dx}{dt} = \frac{d}{dt}\left[\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}m\omega_0^2 x^2\right] + \gamma m\left(\frac{dx}{dt}\right)^2$$

If you know enough differential calculus, you can demonstrate that the equations are the same by taking the differential of $\left[\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}m\omega_0^2 x^2\right]$ and showing that it's equal to $\left[m\frac{dx}{dt}\frac{d^2x}{dt^2} + m\omega_0^2 x\frac{dx}{dt}\right]$.

This doesn't look exactly like our equations for potential and kinetic energy, but remember that $v = \frac{dx}{dt}$ and $k = m\omega_0^2$. Substituting, we get $\left[\frac{1}{2}mv^2 + \frac{1}{2}kx^2\right]$. So the first two terms are just the kinetic and potential energy. Therefore, the work of the forcing function is just the energy lost to the resistance, $\gamma m\left(\frac{dx}{dt}\right)^2$. This is exactly what we'd intuitively expect.

The way we're going to solve this system is to look at the equation for the forced harmonic oscillator with friction, when the force is zero (when the force is removed). We're going to assume that the forcing function is removed when the mass is at the end of one of it's travels (highest or lowest position). We can solve for the general case of removal of the forcing function at any point of the swing, but we're going to take the simple case here.

$$m\frac{d^2x}{dt^2} + \gamma m\frac{dx}{dt} + m\omega_0^2 x = 0$$

The first thing we notice is that we can divide both sides by m, giving

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0$$

Like before, we're going to use an exponential function as a trial solution. We'll use $Ae^{i\alpha t}$.

$$\frac{d^2Ae^{i\alpha t}}{dt^2} + \gamma\frac{dAe^{i\alpha t}}{dt} + \omega_0^2 Ae^{i\alpha t} = 0$$

After taking the differentials, this becomes

$$-\alpha^2 Ae^{i\alpha t} + i\alpha\gamma Ae^{i\alpha t} + \omega_0^2 Ae^{i\alpha t} = 0$$

Then, factoring $Ae^{i\alpha t}$ gives

$$(-\alpha^2 + i\alpha\gamma + \omega_0^2)Ae^{i\alpha t} = 0$$

One solution is if $A = 0$, but that means the system is not oscillating, which is not a very interesting solution. So let's look further. If A is not zero, the only other possibility is

$$-\alpha^2 + i\alpha\gamma + \omega_0^2 = 0$$

This is a simple quadratic equation which can be solved for α using the formula

$$\alpha = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\alpha = \frac{-i\gamma \pm \sqrt{(i\gamma)^2 - 4(-1)(\omega_0^2)}}{2(-1)}$$

$$\alpha = \frac{-i\gamma \pm \sqrt{-\gamma^2 + 4\omega_0^2}}{-2}$$

$$\alpha = \frac{-i\gamma \pm 2\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}}{-2}$$

$$\alpha = i\frac{\gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

To begin with, we're going to assume that γ is small compared to ω_0^2 so the square root is real. Later, we'll examine what happens when γ is large compared to ω_0^2. And just so that I don't have to keep writing the square root, let's define

$$\omega_1 = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

So now, I can write α as

$$\alpha = i\gamma / 2 \pm \omega_1$$

Which is more compact than the previous way of writing the solution. Now, let's substitute back into our assumed solution for α.

$$Ae^{i\alpha t} = Ae^{i(i\gamma/2 \pm \omega_1)t} = Ae^{-\gamma t/2 \pm i\omega_1 t} = Ae^{-\gamma t/2} e^{\pm i\omega_1 t}$$

Note that we have two solutions because of the ± sign. And since we only take the real portion of the solution, our solutions are

$$x_1 = A_1 e^{-\gamma t} \cos \omega_1 t$$

$$x_2 = A_2 e^{-\gamma t} \cos(-\omega_1 t)$$

It turns out that if x_1 and x_2 are solutions to our equation, then x_1 + x2 is also a solution. We can also write this solution as

$$x = e^{-\gamma t/2}\left(A_1 e^{i\omega_1 t} + A_2 e^{-i\omega_1 t}\right)$$

In the general case, A_1 and A_2 can be complex numbers which allow us to accommodate the case where the forcing function is removed when the mass is not at the extreme of a swing.

Since we only take the real portion of the solution, if $A_1 = A_2$, the equation reduces to (note that $\cos(-\omega_1) = \cos\omega_1$)

$$x = e^{-\gamma t/2}\left(2A\cos\omega_1 t\right)$$

We can also show this another way when $A_1 = A_2$, using Euler's equation

$$x = e^{-\gamma t/2}\left(A(e^{i\omega_1 t} + e^{-i\omega_1 t})\right)$$

$$x = e^{-\gamma t/2}\left(\frac{2A(e^{i\omega_1 t} + e^{-i\omega_1 t})}{2}\right)$$

$$x = e^{-\gamma t/2}\left(2A\frac{(e^{i\omega_1 t} + e^{-i\omega_1 t})}{2}\right)$$

Back on page 13, we defined the cosine to be

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \text{Euler's equation}$$

So, substituting

$$x = e^{-\gamma t/2}\left(2A\cos\omega_1 t\right)$$

Remember earlier I stated that this analysis was for the case where γ is small compared to ω_0^2 so that ω_1 is real. This case is known as the *underdamped* case. In the underdamped case, the system continues to oscillate but the amplitude of the oscillations decreases by the exponential factor $e^{-\gamma t/2}$. This is illustrated in Figure 27.

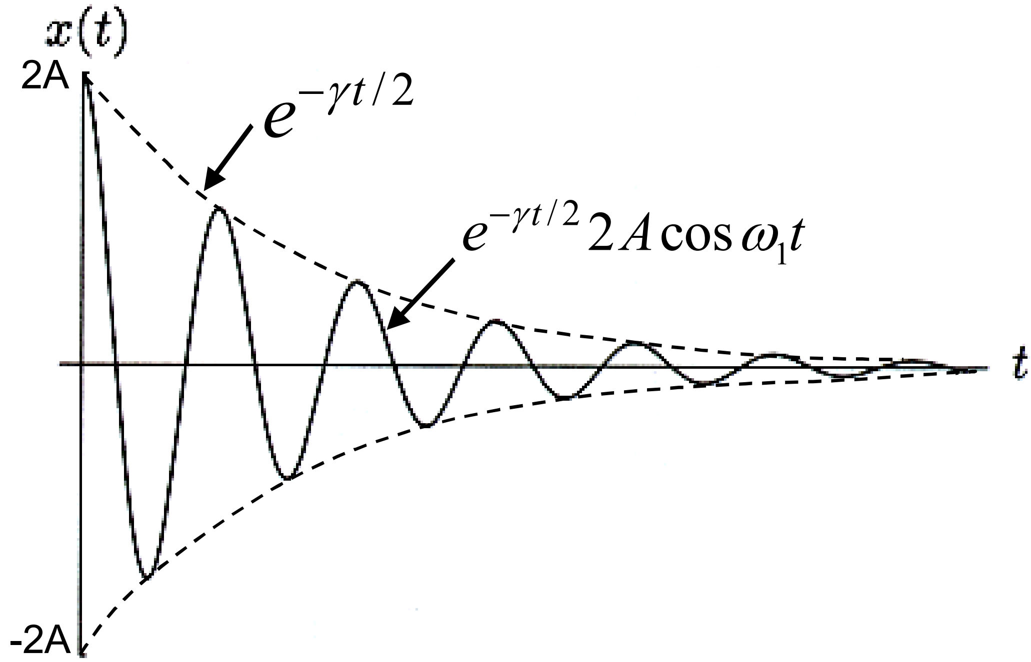

Figure 27: The underdamped case. Note the "envelope" of the cosine function

When γ is just the right size so that ω_1 is zero, we have what is known as the *critically damped* case. Let's examine the mathematics

$$x = e^{-\gamma t/2}\left(A(e^{i\omega_1 t} + e^{-i\omega_1 t})\right)$$

But $\omega_1 = 0$, so

$$x = e^{-\gamma t/2}\left(A(e^{i0t} + e^{-i0t})\right)$$

$$x = e^{-\gamma t/2}\left(A(1+1)\right)$$

$$x = 2Ae^{-\gamma t/2}$$

What happens here is that the mass approaches the rest position in an exponential manner, without going past it. The system stops oscillating as soon as we remove the forcing function. See Figure 28.

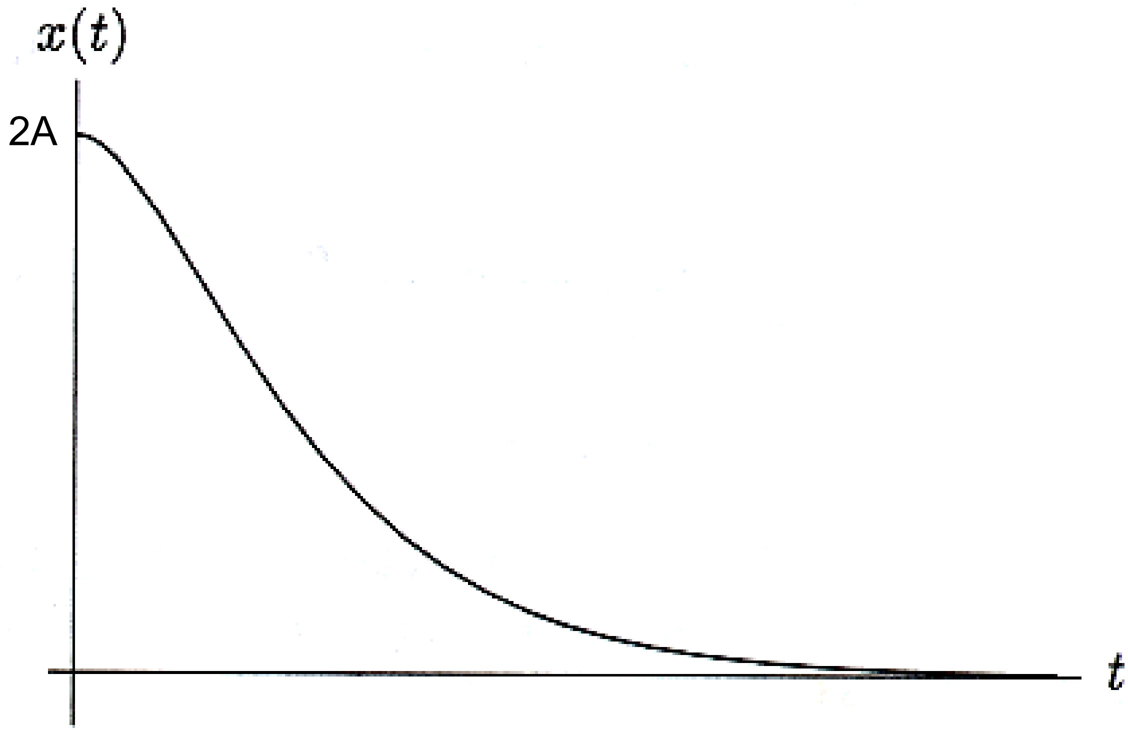

Figure 28: The critically damped case.

The final case, called the *overdamped* case, is when γ is large enough so that $\omega_0^2 - \frac{\gamma^2}{4}$ is negative. When we take the square root, $\omega_1 = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$, we wind up with ω_1 as an imaginary number. Our equation for α becomes

$$\alpha = i\gamma / 2 \pm i\omega_1$$

Which can be simplified to

$$\alpha = i\left(\frac{\gamma}{2} \pm \omega_1\right)$$

Note that even when we take the negative of ω_1, the term $\left(\frac{\gamma}{2} \pm \omega_1\right)$ can never be negative. ω_1 is the result of combining $\frac{\gamma^2}{4}$ and ω_0^2 such that the result is negative. In effect, we are subtracting ω_0^2 from $\frac{\gamma^2}{4}$. Think of it as $-\left(\frac{\gamma^2}{4} - \omega_0^2\right)$ where the term in parentheses must be positive in order to have a negative result. The worse case would be that ω_0^2 would be zero, in which case the square root, ω_1, would be $-\dfrac{\gamma}{2}$, and the term $\left(\dfrac{\gamma}{2} \pm \omega_1\right)$ would be either zero or γ. But having a zero value of ω_0^2 is really not realistic, so we always wind up with a positive term for α. This is very important because if ω_1 were able to exceed $\dfrac{\gamma}{2}$, the term $\left(\dfrac{\gamma}{2} - \omega_1\right)$ would be negative and we'd wind up with a positive exponent when we substituted into our solution. This would give us an increasing exponential, indicating that the amplitude of the swing would increase without end when we removed the forcing function. Not very realistic!!

When we substitute for α back into our original assumed solution, we get

$$Ae^{i\alpha t} = Ae^{i(i\left(\frac{\gamma}{2} \pm \omega_1\right))t} = Ae^{-\left(\frac{\gamma}{2} \pm \omega_1\right)t}$$

So our two solutions are

$$x = A_1 e^{-(\gamma/2 + \omega_1)t}$$

$$x = A_2 e^{-(\gamma/2 - \omega_1)t}$$

And our general solution is

$$x = A_1 e^{-(\gamma/2 + \omega_1)t} + A_2 e^{-(\gamma/2 - \omega_1)t}$$

Which is the sum of two real-valued decaying exponentials. An example of this is shown in Figure 29.

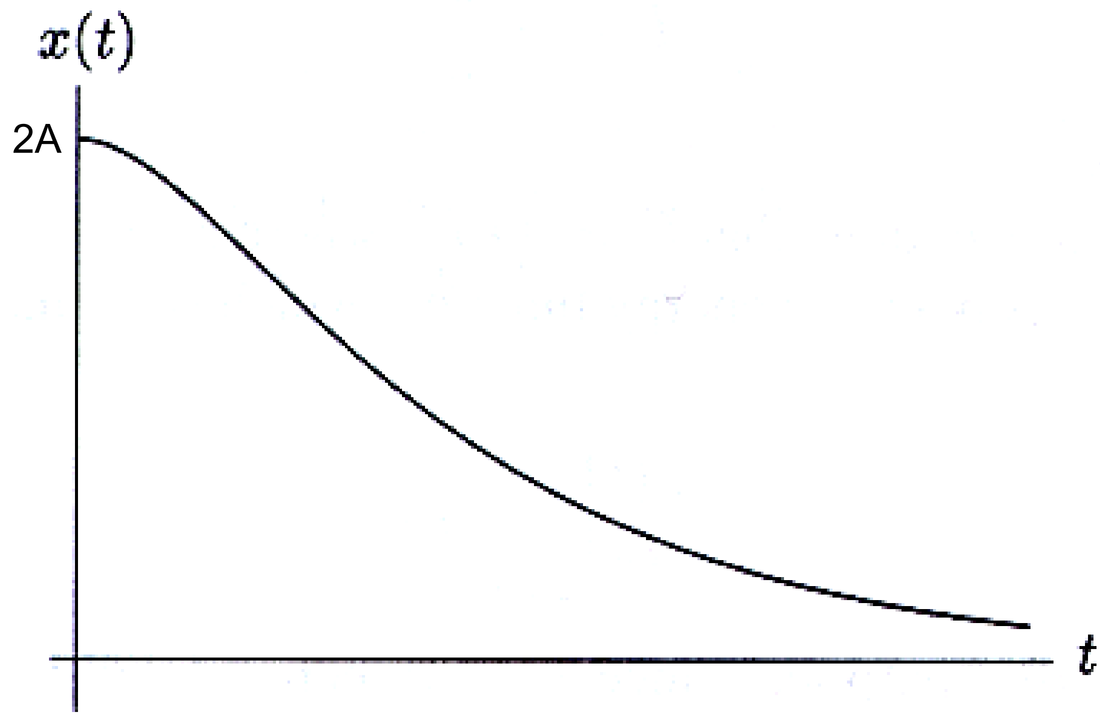

Figure 29: The overdamped case.

Extension to the Electrical Domain

In this paper, I analyzed a mechanical system because mechanical systems are easy for people to visualize. Mechanical systems are things you can put your hands on, or which are easy to visualize from daily experience. But in reality, mechanical systems are generally not as well behaved as these equations assume. Friction is usually not a linear function of velocity and the "stiffness" of the spring is probably not linear over the range of motion. Because of this, mechanical systems are usually analyzed using numerical methods and simulation.

However, there is a domain where these equations work well, and that is the electrical domain. It turns out that each of the characteristics we discussed for mechanical systems have an analog in the electrical domain.

The equivalent of the distance moved, x, is the charge, q. In electrical systems, we usually use current instead of charge, and current is the rate of change of the charge, or dq/dt.

The element in electricity which has inertia is known as inductance. Because of the way an inductor works, it resists changes in the rate of change of the charge. The element which is related to stiffness is capacitance (actually the inverse of capacitance). Capacitance has the property of offering a small opposition to the initial flow of charge, but offering increasing opposition to that flow as time goes by (we're assuming charge flow in a single direction here).

Friction is represented by resistance, in a straightforward fashion.

Other characteristics of the system, such as the resonant frequency and Q have analogs in electrical systems. Table 3 shows how these are related.

General Characteristic	Mechanical property	Electrical property
Independent variable	Time (t)	Time (t)
Dependent variable	Distance (x)	Charge (q)
Inertia	Mass (m)	Inductance (L)
Resistance	Drag coefficient ($c = \gamma m$)	Resistance ($R = \gamma L$)
Stiffness	Spring constant (k)	1/capacitance (1/C)
Resonant frequency	$\omega_0^2 = k / m$	$\omega_0^2 = 1 / LC$
Period	$t_0 = 2\pi\sqrt{m / k}$	$t_0 = 2\pi\sqrt{LC}$
Q (figure of merit)	$Q = \omega_0 / \gamma$	$Q = \omega_0 L / C$

Table 3: The relationship between the characteristics of mechanical systems and electrical systems

If you take the equations we worked out, and substitute the electrical characteristics into them, the solutions will work out just as they did for the mechanical case.

However, that's not the way we usually do things in electrical engineering. The primary difference is that we use current, *I* or *i*, instead of dq/dt. So everywhere there's a dq/dt in the equations we replace it with *I* or *i*. And since we use *I* or *i* for the current, electrical engineers use *j* to represent $\sqrt{-1}$.

Summary

This paper has attempted to give you the basic mathematical understanding necessary to solve differential equations. The actual differential equations solved in the paper were fairly simple – much more complex equations exist. And often, the challenge is in setting up the equation in the first place, rather than in solving it once you have it set up.

If you followed the mathematics developed in this paper, you should be able to follow the mathematics in your university lectures in engineering and physics.

But differential equations are just one part of the engineering /physics curriculum – there are many other challenges which you will encounter. And what I did in this paper was to present a very selected group of mathematics which led to the solution of these specific differential equations. So pay attention in your high school and university mathematics classes. When you encounter other challenges, you'll need other tools. If all you have is a hammer, all problems look like nails. Try to learn all the mathematics, at least to some level.

All through life, you'll continue to be amazed that you encounter problems that relate to something you just studied. The problems were always there, but now you're sensitized to them, and can deal with them. So keep an open mind, and keep learning. It's a lifelong process.

Appendix A – A few miscellaneous equations and solutions

In the footnote on page 15, I asked you to look at two "strange" results from manipulating the equation

$$e^{i\pi/2} = i$$

First, to raise i to the i power, and then to take the natural log of i. In case you had any problems doing it, here's the explanation.

Raising *i* to the *i* power

Let's first address taking i to the i power.

$$i = e^{i\pi/2}$$

$$i^i = (e^{i\pi/2})^i$$

$$i^i = e^{i*i\pi/2}$$

Notice that we have ($i * i$) in the exponent, which is equal to –1. Substituting back into the equation gives

$$i^i = e^{-\pi/2}$$

or

$$i^i = 1/e^{\pi/2}$$

So i raised to the i power is a real number – it's no longer imaginary. Magic, isn't it?

Taking the natural log of *i*

Now, let's take the natural log of i.

$$i = e^{i\pi/2}$$

$$\ln(i) = \ln e^{i\pi/2}$$

Remember that the log of a number to a power is equal to the power times the log. So we now have

$$\ln(i) = i\pi/2 \ln e$$

And since $\ln e = 1$ we have

$$\ln(i) = i\pi/2$$

Taking the log of a negative number

In your high school algebra courses, you were probably told that the log of a negative number is undefined. Well, that's not quite correct. In this section, we'll look at what the log of a negative number really is. We're going to start with Euler's equation from page 12:

$$e^{ix} = \cos x + i \sin x$$

evaluated at $x = \pi$.

$e^{i\pi} = -1$ (if you don't remember this, see page 14)

Now, let's take the natural log of both sides

$$\ln(-1) = \ln(e^{i\pi})$$

$$\ln(-1) = i\pi \ln(e)$$

$$\ln(-1) = i\pi$$

So the log of a negative number is a complex number. Now, your may say, "Wait a minute! All you did was take the natural log of minus one. What if I want the common log of a negative number other than minus one?" Let's explore that. First, we'll look at the natural log of a number greater than –1. Let's take the natural log of –22.

$$-22 = 22 * e^{i\pi}$$

$$\ln(-22) = \ln(22 * e^{i\pi})$$

$$\ln(-22) = \ln(22) + \ln(e^{i\pi})$$

$$\ln(-22) = \ln(22) + i\pi$$

The natural log of 22 is about 3.091, so the natural log of –22 is

$$\ln(-22) = 3.091 + i\pi$$

Now, we need to address the second part of your question. What if we want the common log of a negative number instead of the natural log? First, remember that the natural log is just a log to the base *e*, or

$$\ln(-22) = \log_e(-22)$$

Remember from page **Error! Bookmark not defined.**, to convert logarithms from one base, b, to a new base, x, we divide by $\log_b x$

$$\log_x c = \log_b c / \log_b x$$

Since we want to convert from natural logs to common logs, we divide by $\log_e(10)$, or 2.302585.

$$\mathrm{Log}_{10}(-22) = (3.091 + i\pi)/2.302$$

$$\mathrm{Log}_{10}(-22) = 3.091/2.302 + i\pi/2.302$$

$$\mathrm{Log}_{10}(-22) = 1.342 + i1.364$$

Since π is just a number, I divided it by 2.302 to get 1.364. But most of the time, I'd choose to keep the π factor and show the result as

$$\mathrm{Log}_{10}(-22) = 1.342 + i\pi/2.302$$

Of course, we're not limited to common logs. Using the same technique, you can obtain the log of a negative number to any base.

Using $Ae^{i\omega t}$ as a solution to our differential equation

The other equation I didn't work out is the solution of the differential equation with a constant factor, $Ae^{i\omega t}$ on page 48. Let's take our original differential equation and solve it with the assumed solution $x = Ae^{i\omega t}$.

$$d^2x/dt^2 = -(k/m)x$$

Assume a solution of $x = Ae^{i\omega t}$

$$d^2(Ae^{i\omega t})/dt^2 = -(k/m)\,Ae^{i\omega t}$$

Take the first differential

$$d(Ae^{i\omega t})/dt = i\omega A\, e^{i\omega t}$$

To get to the second differential, we take the differential of the first differential.

$$d(i\omega A\, e^{i\omega t})/dt = (i\omega)^2\, Ae^{i\omega t}$$

or

$$d^2(Ae^{i\omega t})/dt^2 = -\omega^2\, Ae^{i\omega t}$$

So our original equation

$$d^2(Ae^{i\omega t})/dt^2 = -(k/m)\, Ae^{i\omega t}$$

becomes

$$-\omega^2\, Ae^{i\omega t} = -(k/m)\, Ae^{i\omega t}$$

This time, we can divide both sides by $Ae^{i\omega t}$, leaving

$$-\omega^2 = -k/m$$

Indicating that $Ae^{i\omega t}$ is also a valid solution of our differential equation. So any constant times our original solution of $e^{i\omega t}$ is also a solution. The correct value of A is determined from the initial conditions – in this case, how far we pull the weight down.

Showing that the width of the resonance curve is γ

On page 58, I said that the width of the resonance curve, half way to the peak, was γ. This section contains the mathematics which demonstrates this. We start with

$$r^2 \approx \frac{1}{4m^2\omega_0^2[(\omega_0-\omega)^2-\gamma^2/4]}$$

The peak occurs when $\omega = \omega_0$, causing the equation to reduce to

$$r^2 \approx \frac{1}{m^2\omega_0^2\gamma^2}$$

One half the peak value will be equal to

$$r^2 \approx \frac{1}{2m^2\omega_0^2\gamma^2}$$

Now we can equate

$$2m^2\omega_0^2\gamma^2 = 4m^2\omega_0^2[(\omega_0-\omega^2)^2-\gamma^2/4]$$

And solve for γ. Start by canceling terms.

$$\gamma^2 = 2[(\omega_0-\omega^2)^2-\gamma^2/4]$$

$$\gamma^2/2 = (\omega_0-\omega^2)^2-\gamma^2/4$$

$$\frac{2\gamma^2}{4}+\frac{\gamma^2}{4} = (\omega_0-\omega^2)^2$$

$$\frac{\gamma^2}{4} = (\omega_0-\omega^2)^2$$

$$\frac{\gamma}{2} = (\omega_0 - \omega^2)$$

$$\gamma = 2(\omega_0 - \omega^2)$$

So γ is equal to twice the difference between ω_0 and ω at the point where the curve is half the height of the peak of the resonance curve, which is the width of the resonance curve at that point.

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Appendix B – Further discussion of complex numbers

The cube root of 8

On page 15, I gave you the three cube roots of 8. You may wonder how to calculate these roots. While there's a good, valid mathematical technique to do this (de Moivre's technique[15]), you have to remember that I'm an engineer and look for pragmatic, easy solutions to problems. Now, the fundamental theorem of algebra says that there are *n* roots of an n^{th} degree equation. What the fundamental theorem of algebra doesn't say, but which is true for simple equations of the form $x^n = k$, is *that those roots are evenly spaced around the complex number plane*[16]. Additionally, complex roots always come in pairs which are the complex conjugate of each other[17] – meaning that the complex roots differ only by the sign of the imaginary part. So if one root is $2 + i3$, the other root will be $2 - i3$. This means that you only have to compute the complex roots which exist above the x-axis. Once you have these roots, you can write the rest of the complex roots by inspection.

As a short aside, since complex roots come in pairs, an equation of an odd power (like x^3, x^5, or x^7) must always have a real root. For example, a cubic equation must have either one or three real roots. If it has a complex root, it must have two.

Let's look at a simple equation first:

$$x^2 = 4$$

$$x = \pm 2$$

Now, let's look at these solutions on the complex number plane.

[15] Named for French mathematician Abraham de Moivre (1667 – 1754)

[16] This is true for these simple equations only. If the equation was more complex, like $x^3 + 6x - 20 = 0$, the roots would not be evenly spaced around the complex number plane, nor would the radius vector for each solution be the same length. The roots of this $x^3 + 6x - 20 = 0$ equation are 2, $-1+3i$, and $-1-3i$.

[17] Technically, this is only true when the coefficients are real numbers, but you'll probably never see an equation with complex coefficients except in a mathematics class. An example of an equation with a complex coefficient is $ix^3 + 2x + 10 = 0$

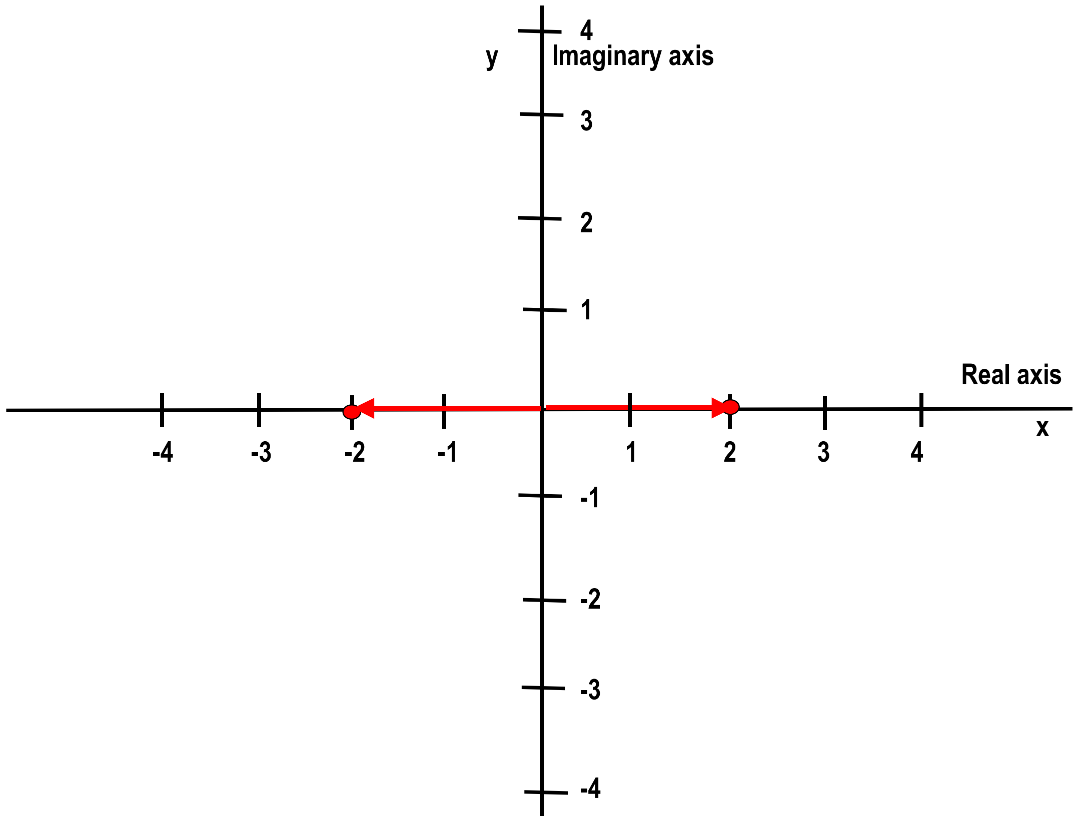

Figure 30: The two solutions for $x^2 = 4$, shown on the complex number plane.

What we see is that the two solutions are symmetrical around the origin, one at +2 and the other at –2. This solution has no complex roots.

Let's now look at the complex number plane for the cube root of 8.

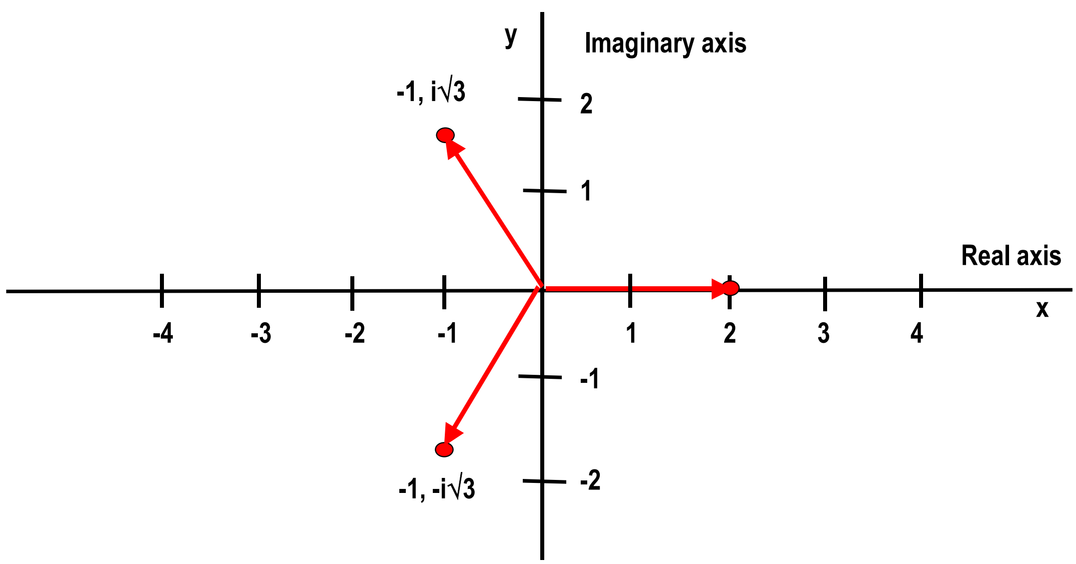

Figure 31: The three solutions for $x^3 = 8$, shown on the complex number plane.

Note that the complex roots are complex conjugates of each other (they differ only in the sign of the imaginary part). Note, also, that the solutions are all of length 2, and they are evenly distributed around the origin. Remember that the length of the radius vector is given by

$$r^2 = a^2 + b^2$$

Where *a* is the real number associated with the real term and *b* is the real number associated with the imaginary term.

$$r^2 = 1^2 + \sqrt{3}^2$$

$$r^2 = 4$$

$$r = 2$$

So the cube root of 8 is three values of 2, and they are evenly distributed around the complex number plane.

But how do we compute the exact values, such as the $-1 + i\sqrt{3}$? Well, we know that there are 360 degrees in a circle (or 2π radians). Since the solutions are evenly distributed around the origin, they must be 120 degrees apart (360/3). This means they must form an angle of 60 degrees with the real axis. Since we know the length of the vector is 2, and the angle is 60 degrees, we can compute the real value from the cosine multiplied by 2. Since the cosine of 60 degrees is 0.5, the real part is 1. Since it's on the left hand side of the y-axis, it's negative.

Once we know the real part, we can compute the imaginary part because $r^2 = a^2 + b^2$. And, of course, that's why the imaginary part is shown as $\sqrt{3}$, because $a^2 + b^2$ must equal 2^2 and that can only happen if $b = \sqrt{3}$. So you don't need some complex mathematical equation (which you can never remember), you just need to know that the roots are evenly spaced around the complex number plane. From that, and a little ingenuity, you can determine all the roots.

But, you might say, "How does $(-1 + i\sqrt{3})^3$ equal 8? Look at that answer! How can that possibly be a cube root of 8?" Let's compute the cube and check the solution. We can compute the cube by repeated multiplication.

$$x^2 = x * x = (-1 + i\sqrt{3}) * (-1 + i\sqrt{3})$$

$$x^2 = +1 - i\sqrt{3} - i\sqrt{3} - 3$$

$$x^2 = -2 - i2\sqrt{3}$$

$$x^3 = x^2 * x = (-2 - i2\sqrt{3}) * (-1 + i\sqrt{3})$$

$$x^3 = +2 + i2\sqrt{3} - i2\sqrt{3} + 2*3$$

Note that the two imaginary terms cancel, leaving only real terms.

$$x^3 = 2 + 6 = 8$$

Amazing, isn't it? I'll let you check the other root, $(-1 - i\sqrt{3})$, on your own.

Now that you know this, let's try another problem. What is the fourth root of 16 ($x^4 = 16$)? You know that one root is 2, but what are the other three roots? Remember that the roots are evenly distributed around the complex number plane and the complex roots always appear in conjugate pairs. Draw the solutions on the complex number plane below and write your answers here

____________________________________. I've indicated the easy answer, (2, *i*0), on the figure. The solution is on the next page.

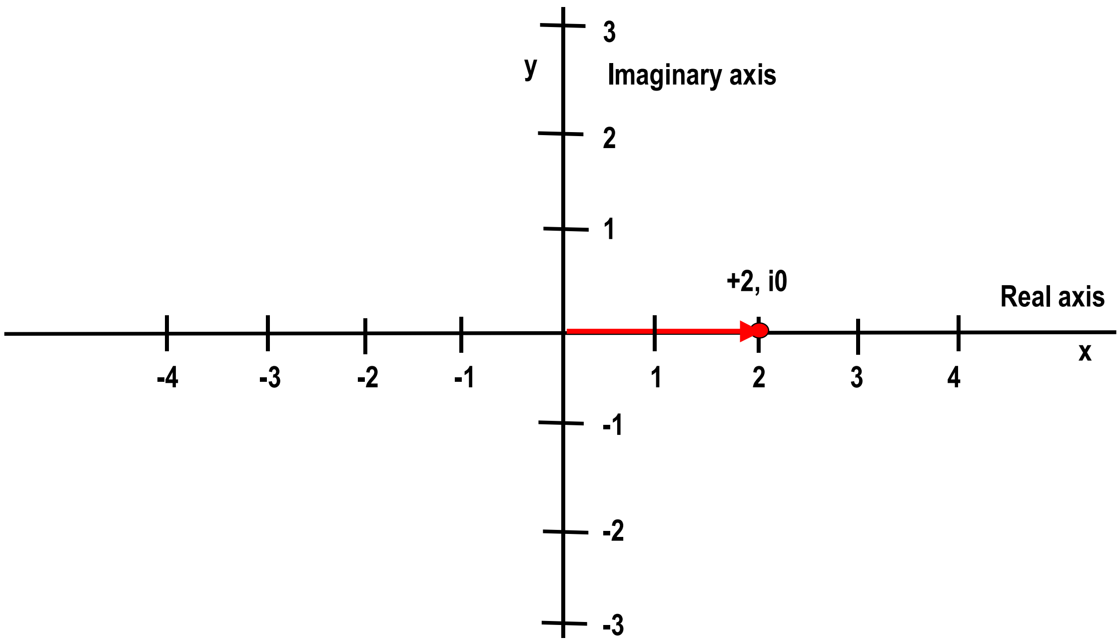

Figure 32: The complex number plane with one solution for $x^4 = 16$. Draw the other three solutions on this diagram.

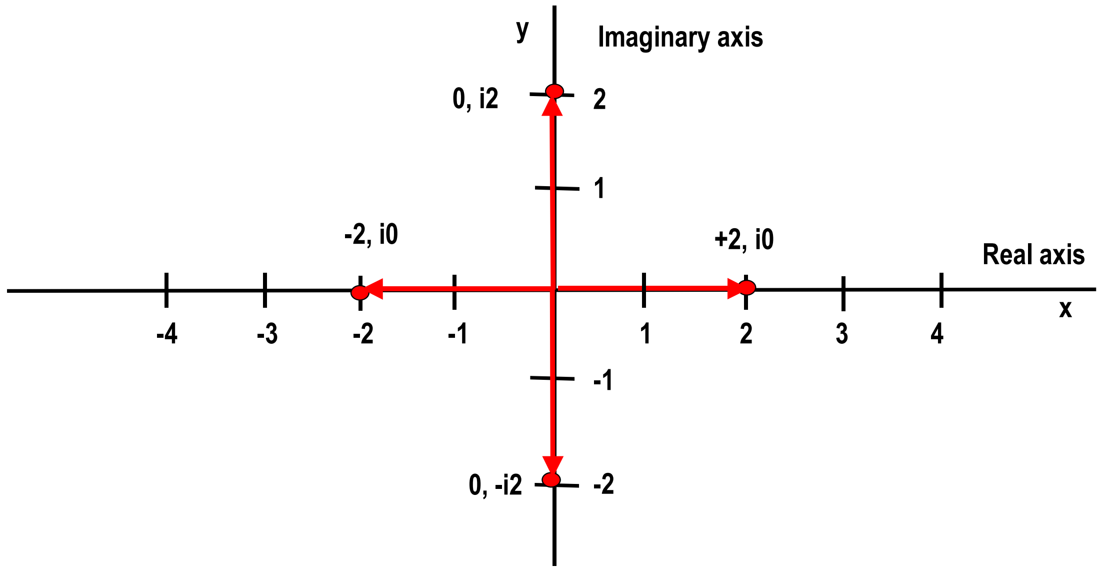

Figure 33: The complex number plane with the four solutions for $x^4 = 16$.

So here's your solution: (+2, i0), (0, i2), (–2, i0), and (0, –i2). The length of each radius vector is 2.

An interesting related problem is to find the nth roots of 1, for example, the fifth root of 1.

$$y = \sqrt[5]{1}$$

The primary root is, of course 1, but there are four other roots evenly spaced around the complex number plane. Since there are 2π radians in a circle, the roots will be evenly spaced 2π /5 radians apart (72 degrees). The second root will be cos(2π/5), $i$sin(2π /5) or 0.309, i0.951. The next root will be at 180 degrees minus 2 times 72 degrees (which is 36 degrees or π –2*2π /5). The value will be cos(π –4π /5), $i$sin(π –4π /5). Since the solution is in the second quadrant, the real component will be negative, giving –0.809, i0.588. Since the complex roots come in conjugate pairs, the four solutions are (1, i0), (0.309, i0.951), (0.309, –i0.951), (–0.809, i0.588), and (–0.809, –i0.588).

Now you can amaze your friends by solving simple cubic and fourth degree equations in your head, simply by using your knowledge of how the solutions must be arranged around the complex number plane.

Raising a Complex Number to a Real Power (including taking the root of a complex number)

In the section on complex numbers, I showed you how to add, subtract, multiply, and divide complex numbers. I also spent quite a bit of time discussing how to raise a real number to a complex power (such as e^{ix}). I did not show you how to take a complex number to a power. Since a power can be fractional, this explanation will include how to take the square root (or cube root, etc.) of a complex number. This explanation is graphical (visual). Some people understand better when they can visualize how a problem is solved. Later, I give a more generalized algebraic solution, which includes complex powers.

Starting from the basics, complex numbers are represented on the complex number plane, where the x-axis represents the real numbers and the y-axis represents the imaginary numbers, as shown in Figure 34.

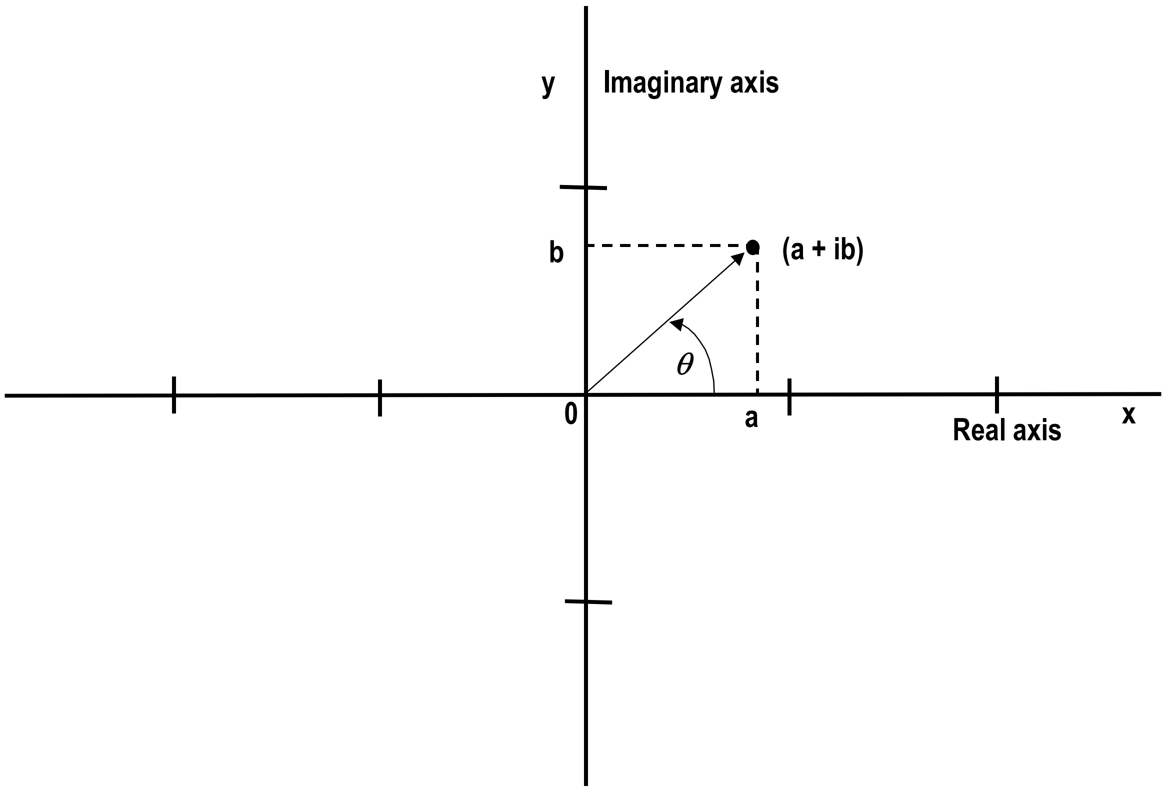

Figure 34: The complex plane.

We can represent a complex number in Cartesian coordinates, in this example (a, b), or in polar coordinates where we give the length of the radius vector and the angle, θ. We can convert from one representation to the other by looking at the radius vector as the hypotenuse of a triangle, with the other sides being of length a and b. The length of the radius vector is given by[18]

$$r^2 = a^2 + b^2$$

or

$$r = \sqrt{a^2 + b^2}$$

The angle θ can be computed from

$$\tan \theta = b/a$$

or

$$\theta = \tan^{-1}(b/a)$$

This can be written as

[18] We can also look at this as multiplication of a complex number by its conjugate. So if $z = a + ib$, the modulus, or length, can be calculated by z times z conjugate, or $(a + ib)(a – ib)$.

$$a + ib = \sqrt{a^2 + b^2} \ \angle \tan^{-1}(b/a)$$

The term $\sqrt{a^2 + b^2}$ is the length of the radius vector and the angle θ is represented by $\angle \tan^{-1}(b/a)$.

Note that when a is a negative number, the resulting angle will be $\pi + \tan^{-1}(b/a)$ rather than $\tan^{-1}(b/a)$. This is because the tangent has a period of π, rather than 2π like the sine and cosine.

Taking a complex number to a power is equivalent to

$$(a + ib)^n = (\sqrt{a^2 + b^2})^n \quad n * \angle \tan^{-1}(b/a)$$

That is, the length of the radius vector is taken to the power, and the angle is multiplied by the power. This works for both powers and roots – after all, a root is just a fractional power.

Let's do a couple of sample problems, one taking a complex number to a power and the other taking a complex number to a root. Let's use the complex number $(2.52 + i1.68)$. First, we'll raise it to the 15^{th} power and then we'll take the 9^{th} root.

$$(2.52 + i1.68)^{15} = (\sqrt{2.52^2 + 1.68^2})^{15} \quad 15 * \angle \tan^{-1}(2.52/1.68)$$

$$(2.52 + i1.68)^{15} = 3.028663^{15} \quad 15 * \angle \tan^{-1}(1.5)$$

$$(2.52 + i1.68)^{15} = 3.028663^{15} \quad 15 * \angle 0.588003 \text{ radians (or about 33.69 degrees)}$$

$$(2.52 + i1.68)^{15} = 16{,}548{,}721 \quad \angle 8.820039 \text{ radians (or about 505.35 degrees)}$$

Now 8.820039 radians is greater than 2π radians so we need to subtract 2π radians from the angle repeatedly until the value is less than 2π radians. It turns out that we only have to subtract 2π once, giving an angle of 2.536854 radians. So our result is

$$(2.52 + i1.68)^{15} = 16{,}548{,}721 \quad \angle 2.536854 \text{ radians (or about 145.351 degrees)}$$

If we want to convert this back to the standard form for complex numbers, we can use the equation

$$y = r(\cos\theta + i\sin\theta)$$

$$y = 16{,}548{,}721 (\cos 2.536854 + i \sin 2.536854)$$

$$y = 16{,}548{,}721 (-0.82265 + i\,0.568547)$$

$$y = -13{,}613{,}815 + i\,9{,}408{,}731$$

Now, let's take the 9^{th} root of the same complex number.

$$(2.52 + i1.68)^{1/9} = (\sqrt{2.52^2 + 1.68^2})^{1/9} \quad 1/9 * \angle \tan^{-1}(2.52/1.68)$$

$(2.52 + i1.68)^{1/9} = 3.028663^{1/9} \quad 1/9 * \angle \tan^{-1}(1.5)$

$(2.52 + i1.68)^{1/9} = 3.028663^{1/9} \quad 1/9 * \angle 0.588003$ radians (or about 33.69 degrees)

$(2.52 + i1.68)^{1/9} = 1.131025 \quad \angle 0.0653336$ radians (or about 3.743331 degrees)

Converting to standard form complex numbers,

$y = r (\cos \theta + i \sin \theta)$

$y = 1.131025 (\cos 0.0653336 + i \sin 0.0653336)$

$y = 1.131025 (.997867 + i\, 0.073841)$

$y = 1.128612 + i\, 0.073841$

Raising a Complex Number to a Complex Power

It should come as no surprise that is also possible to raise a complex number to a complex power. The approach is a bit different from that described above but it's not really difficult. In fact, this approach is general and can be used with any combination of real and complex numbers – real number to a real power, complex number to a real power, real number to a complex power, and complex number to a complex power. The reason it works is that a real number is just a special case of a complex number, i.e., when the imaginary part is zero.

Let's examine how to do the mathematics. I'll start by discussing the general approach including giving you the generalized equations. Then I'll work some example problems.

Let's take the complex number a + *i*b and raise it to the power c + *i*d.

$y = (a + ib)^{c + id}$

To begin to solve this equation, we start with Euler's equation:

$e^{ix} = \cos x + i \sin x$

As shown here, Euler's equation assumes a radius vector of length one. However, in the general case, the radius vector is equal to:

$$r = \pm\sqrt{a^2 + b^2}$$

As long as r is not zero, we can represent it as

$$r = e^k$$

with $k = \ln(r)$. So the general case[19] of Euler's equation is:

$$e^k e^{ix} = r(\cos x + i \sin x)$$

$$e^{k + ix} = r(\cos x + i \sin x)$$

To demonstrate that this is correct, let's substitute for the cosine and sine, remembering that the sine is equal to "opposite over hypotenuse" and the cosine is equal to "adjacent over hypotenuse." r is the hypotenuse, b is the opposite, and a is the adjacent. Substituting,

$$e^{k + ix} = r(a/r + i\, b/r)$$

$$e^{k + ix} = (a + i\, b)$$

We can find the angle x by

$$\sin x = b/r \qquad (\text{or } \cos x = a/r)$$

$$x = \sin^{-1}(b/r)$$

Now, taking both sides to the $c + id$ power.

$$(e^{k + ix})^{c + id} = (a + i\, b)^{c + id}$$

$$e^{(ck-dx) + i(cx+dk)} = (a + i\, b)^{c + id}$$

$$e^{(ck-dx)}\, e^{i(cx+dk)} = (a + i\, b)^{c + id}$$

$$e^{(ck-dx)}\, e^{i(cx+dk)} = (a + i\, b)^{c + id}$$

$e^{(ck-dx)}$ is a real number. And we can express $e^{i(cx+dk)}$ using Euler's equation.

$$e^{(ck-dx)}\, e^{i(cx+dk)} = e^{(ck-dx)}(\cos(cx+dk) + i \sin(cx+dk))$$

With actual numbers, this can be computed to result in a complex number of the form $a + ib$. Let's take a specific example, using $(2.52 + i1.68)$ raised to the $(3.88 + i2.62)$ power. Let's start by computing r.

$$r = \sqrt{a^2 + b^2}$$

$$r = (\sqrt{2.52^2 + 1.68^2})$$

$$r = 3.028663$$

$$k = 1.10812$$

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[19] To handle the negative result of the square root, we can use $-r = -e^k$. When we substitute into Euler's equation, we introduce a negative on both sides, which cancels out: $-e^k e^{ix} = -r(\cos x + i \sin x)$. So we do take care of both roots of the square root of r.

The angle *x* can be computed by the sine or cosine function, since we have already computed *r*.

$\cos x = a/r$

$\cos x = 2.52/3.028663$

$\cos x = 0.83205$

$x = \cos^{-1} 0.83205$

$x = 0.588003$ radians (or about 33.69 degrees)

Note that I used the same complex number, $(2.52 + i1.68)$, as in the previous example problem and we calculated the same angle in the complex plane – which is what should happen.

We can now substitute back into the equation we derived earlier,

$$e^{(ck-dx)}\, e^{i(cx+dk)} = e^{(ck-dx)}(\cos(cx+dk) + i \sin(cx+dk))$$

with the values:

a = 2.52
b = 1.68
c = 3.88
d = 2.62
r = 3.028663
k = 1.10812
x = 0.58803 (radians)

$$e^{(ck-dx)}\, e^{i(cx+dk)} = e^{(4.2995-1.54057)}(\cos(2.28745+2.90328) + i \sin(2.28745+2.90328))$$

$$e^{(ck-dx)}\, e^{i(cx+dk)} = e^{(2.75894)}(\cos(5.18473) + i \sin(5.18473))$$

$$e^{(ck-dx)}\, e^{i(cx+dk)} = 15.78316(0.45497 - i\, 0.89057)$$

$$e^{(ck-dx)}\, e^{i(cx+dk)} = (7.180871 - i\, 14.055011)$$

And for extra points, let's calculate i^i and 1^1 using the above equation. Back on page 69 we calculated the value of i^i to be $e^{-\pi/2}$. Now, let's see if our general equation gives the same result.

$$e^{(ck-dx)}\, e^{i(cx+dk)} = e^{(ck-dx)}(\cos(cx+dk) + i \sin(cx+dk))$$

a = 0
b = 1
c = 0
d = 1
r = 1
k = 0
x = 1.5708 (π/2 radians)

$e^{(ck-dx)}\ e^{i(cx+dk)} = e^{(0-\pi/2)}\ (\cos (0 + 0) + i \sin (0 + 0))$

$e^{(ck-dx)}\ e^{i(cx+dk)} = 0.20788 * (1+ i0)$

$e^{(ck-dx)}\ e^{i(cx+dk)} = 0.20788$

To check this, compute the value of $e^{-\pi/2}$

$e^{-\pi/2} = 0.20788$

Now, let's compute the value of 1^1

a = 1
b = 0
c = 1
d = 0
r = 1
k = 0
x = 0 (radians)

$e^{(ck-dx)}\ e^{i(cx+dk)} = e^{(0-0)}(\cos (0+0) + i \sin (0+0))$

$e^{(ck-dx)}\ e^{i(cx+dk)} = 1 * (1+ i0)$

$e^{(ck-dx)}\ e^{i(cx+dk)} = 1$

The equation also works to take a real number to a fractional power. For our last example, let's take the square root of 4.

a = 4
b = 0
c = 0.5
d = 0
r = 4
k = 1.3863
x = 0 (radians)

$e^{(ck-dx)}\ e^{i(cx+dk)} = e^{(0.693147-0)}(\cos (0+0) + i \sin (0+0))$

$e^{(ck-dx)}\ e^{i(cx+dk)} =2 * (1+ i0)$

$e^{(ck-dx)}\ e^{i(cx+dk)} = 2$

I guess you could say that this is a complex way to compute the square root of 4.

This equation works for any real or complex number to any real or complex power. But suppose the complex power is a complex root, such as $1/i$? The equation still works but we just have to see how to make $c + id = 1/i$.

The easiest way to do this is to remember the trick we use to divide two complex numbers – we multiply by 1 in the form of the conjugate of the denominator. Since *i* can be expressed as 0 + *i,* the conjugate is 0 – *i.*

$$\frac{(1+0i)}{(0+i)} * \frac{(0-i)}{(0-i)}$$

Now, multiply out.

$$\frac{-i}{-i^2}$$

or

$$\frac{-i}{1}$$

So we have $c = 0$ and $d = -1$.

So we can raise a complex number to a complex root by some simple algebraic manipulation. The equation does indeed work for any combination of real and complex numbers.

Let's check this by taking the i^{th} root of –1. Since we know that for 1/*i*, $c = 0$ and $d = -1$, our table looks as follows:

a = –1
b = 0
c = 0
d = –1
r = 1
k = 0
x = π (radians)

$$e^{(ck-dx)}\ e^{i(cx+dk)} = e^{(0\,+\,\pi)}(\cos(0+0) + i\sin(0+0))$$

$$e^{(ck-dx)}\ e^{i(cx+dk)} = 23.141 * (1+ i0)$$

$$e^{(ck-dx)}\ e^{i(cx+dk)} = 23.141$$

So the i^{th} root of –1 is a positive real number, much larger than 1. But suppose that we took the i^{th} root of +1. If you substitute into the equations above, you'll find that the result is 1.

The results of complex number calculations are not intuitive because the calculations involve rotations in the complex number plane. You have to do the calculations to see what the results are.

Trigonometric Functions of Complex Numbers

Earlier, we talked about expressing rotation angles in radians, and that there are 2π radians in a circle. But what if the number of radians is a complex number? Does that make sense? And can we calculate the sine of a "complex angle"? Let's find out!

Let's start by examining the sine of a complex number $(a + ib)$

$$\sin(a + ib) = ?$$

We start by using the identity

$$\sin(\theta + \phi) = \sin \theta \cos \phi + \cos \theta \sin \phi$$

Now, using $\theta = a$ and $\phi = ib$, we have

$$\sin(a + ib) = \sin(a) \cos(ib) + \cos(a) \sin(ib)$$

But how do we evaluate $\sin(ib)$ and $\cos(ib)$? Earlier, on page 13, I gave you the equations for sine and cosine in exponential form, which I repeat below.

$$\sin x = (e^{ix} - e^{-ix})/2i$$

$$\cos x = (e^{ix} + e^{-ix})/2$$

The sine and cosine are functions of a circle, which has an equation of the form $x^2 + y^2 = 1$. There's another form of sine and cosine, called hyperbolic sine and cosine, which arise from analysis of hyperbolas which have the form $x^2 - y^2 = 1$. The exponential form equations for the hyperbolic sine, abbreviated sinh, and the hyperbolic cosine, abbreviated cosh, are as follows

$$\sinh x = (e^{x} - e^{-x})/2$$

$$\cosh x = (e^{x} + e^{-x})/2$$

If we use the exponential form equations for the sine and cosine and substitute *ix* for *x* everywhere in the equation, we find that

$$\sin(ix) = i \sinh(x)$$

$$\cos(ix) = \cosh(x)$$

This, then, leads to the equation

$$\sin(a + ib) = \sin(a) \cosh(b) + \cos(a)\, i \sinh(b)$$

or

The sinh and cosh

I'm going to quickly show that
$\sin(ix) = i \sinh(x)$

We have the equation for the sine to the left of this box. Let's substitute for *x* with *ix*.

$$\sin(ix) = (e^{i(ix)} - e^{-i(ix)})/2i$$

$$\sin(ix) = (e^{-x} - e^{x})/2i$$

Now, multiply by one in the form of *i/i*.

$$\sin(ix) = i(e^{-x} - e^{x})/(-2)$$

Multiply by $(-1)/(-1)$.

$$\sin(ix) = i(e^{x} - e^{-x})/2$$

So

$$\sin(ix) = i(e^{x} - e^{-x})/2 = i \sinh(x)$$

Demonstrating that

$$\cos(ix) = \cosh(x)$$

is straightforward and will not be shown here.

$$\sin(a + ib) = \sin(a)\cosh(b) + i\cos(a)\sinh(b)$$

For the cosine of a complex number, we start with the identity

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

Going through the same analysis as given above, we wind up with

$$\cos(a + ib) = \cos(a)\cosh(b) - i\sin(a)\sinh(b)$$

Of course, once we have the sine and cosine, we can construct any other trigonometric function, such as the tangent, cotangent, secant, or cosecant.

The sine or cosine of a real number has a range between +1 and –1. The sine or cosine of a complex number is not limited to the same range, primarily because the hyperbolic sine and cosine extend to infinity. Let's do a few examples. First, let's take the sine of our favorite complex number ($2.52 + i1.68$).

$$\sin(a + ib) = \sin(a)\cosh(b) + i\cos(a)\sinh(b)$$

$$\sin(2.52 + i1.68) = \sin(2.52)\cosh(1.68) + i\cos(2.52)\sinh(1.68)$$

$$\sin(2.52 + i1.68) = 0.5823 * 2.77596 + i\,(-0.81295) * 2.58959$$

$$\sin(2.52 + i1.68) = 1.616529 - i\,2.10521$$

Now, suppose that we ask a different question. What complex number has a sine of 10? We want the imaginary part to be zero, which means that cos(a) must be zero. Therefore, *a* must equal π. Since $\sin(\pi) = 1$, we can now find *b* by asking, "The cosh of what number equals 10?" It turns out to be about 2.99322285.

So we can say

$$\sin(\pi + i\,2.99322285) = 10$$

So we see that the sine and cosine can take on any value, provided we allow a complex number of radians.

Bibliography

Dunham, William, *Euler, The Master of Us All*. The Mathematical Association of America, 1999.

Feynman, Richard P., Robert B. Leighton, and Matthew Sands, *The Feynman Lectures on Physics, Volume 1*. Addison-Wesley Publishing, Reading, MA, 1963. (See especially chapter 21, "The Harmonic Oscillator" and chapter 22, "Algebra". Feynman comments in a later volume that he was disappointed with the test results of the students who took his introductory physics class at Caltech. Read this book thinking about how you would have fared if you had walked into Feynman's lectures right out of high school.)

Gullberg, Jan, *Mathematics: From the Birth of Numbers*. W. W. Norton, 1997. (Excellent book, recommended to anyone with an interest in mathematics.)

Maor, Eli, *e: The Story of a Number*. Princeton University Press, 1994. (Good book on *e*, understandable to anyone with high school mathematics.)

Nahin, Paul J., *An Imaginary Tale. The Story of* $\sqrt{-1}$. Princeton University Press, 1998. (This book is not a good introduction to imaginary numbers – it just requires too much knowledge of mathematics to be useful to a high school student.)