Working with Logarithms

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After I wrote the paper "Differential Equations for High School Students", I had it reviewed by several friends who are knowledgeable in differential equations and in mathematics in general. They gave me a number of good suggestions relating to the paper, most of which have already been incorporated into the paper. One of their suggestions was to remove the sections on logarithms because those sections were overly detailed, difficult to understand, and distracted from the main thrust of the paper. They also felt that most high school students would be familiar enough with logarithms that they could understand the use of logarithms in the later section of the paper.

I've taken their suggestions, which is why you're reading this paper.

I put this material in the original paper because I found it very interesting the way Briggs and others calculated logarithms before there were infinite series for the logarithm and before electronic calculators. People are clever, a statement which is true today as well back in the days when Briggs was first calculating the value of common logarithms, or when we were building the pyramids, for that matter.

For those of you who make it through this material, I hope you find it as interesting as I did.

Logarithms

It's difficult for us, with our personal computers and hand-held calculators, to understand what a tremendous achievement the development of logarithms really was. Prior to logarithms, calculations were extremely laborious, especially for people such as astronomers who needed to multiply and divide very large numbers. With logarithms, multiplication of two numbers becomes an addition, and division becomes a subtraction (as will be explained later in this section). Addition and subtraction are much easier than multiplication and division, and less prone to error.

Even after logarithms, calculations were tedious but they were a lot less so. Logarithms were commonly used until at least the mid 1970's when low cost hand held calculators became popular.

One common way of using logarithms was in the form of a slide rule, an accessory which unmistakably identified the university engineering students. Figure 1 and Figure 2 are pictures of the author's slide rule, used during his university studies.

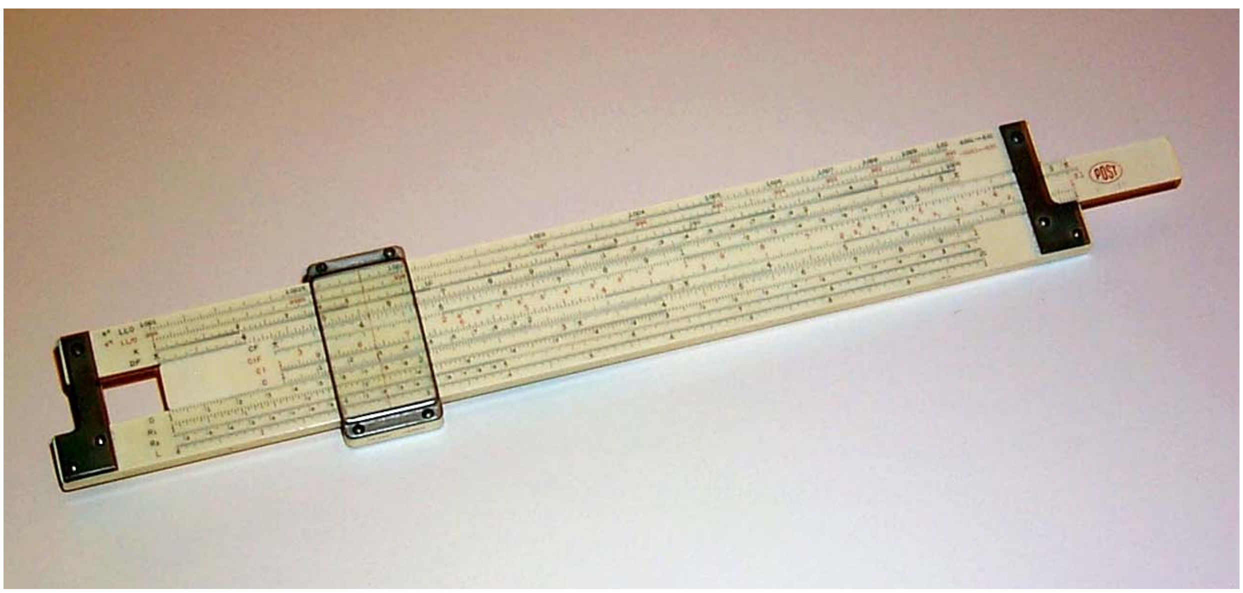

Figure 1: A slide rule uses logarithmic scales to perform multiplication and division. Prior to the availability of low-cost calculators, this was the most popular calculating tool used by engineers and scientists.

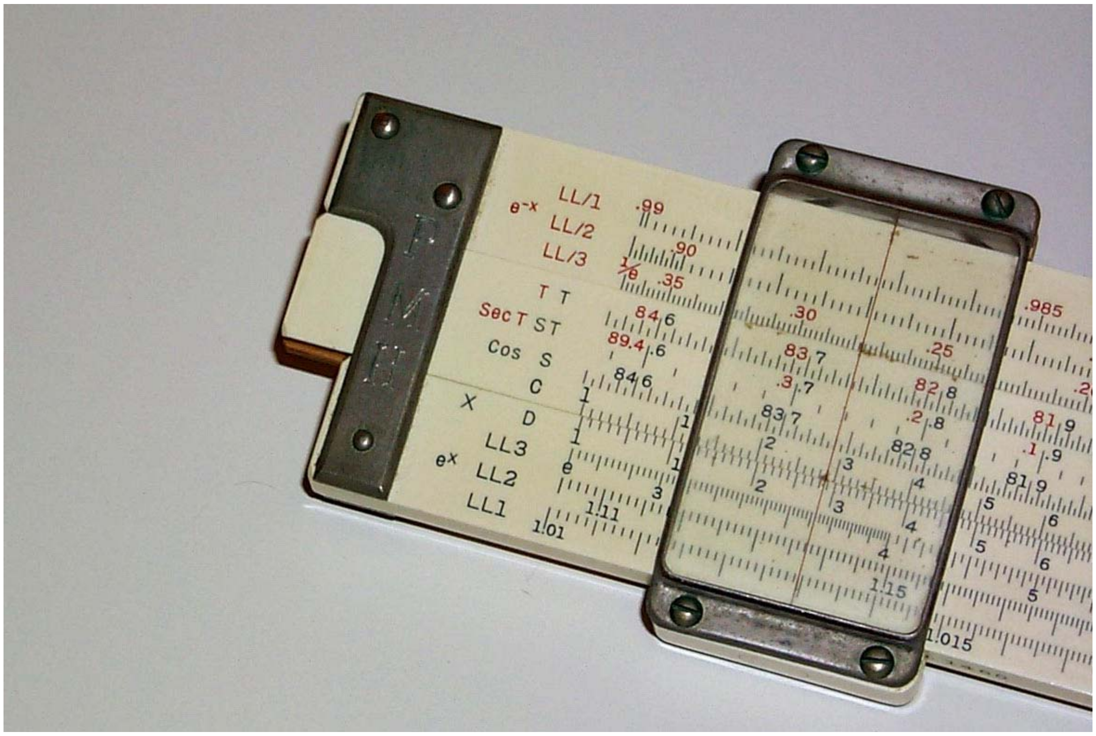

Figure 2: Most engineering students used the same brand of slide rule. But each student adjusted their slide rule differently, some with the slide very loose while others liked the slide tight. When working together on a problem, sometimes we'd lose track of which slide rule was which. To identify my slide rule, I had my initials (PMH) engraved on the rule. I was a real geek – and maybe I still am!

In discussing logarithms, I'm going to first discuss how logarithms work. After that, I'll discuss how logarithms were calculated in the 17th century.

Working with Logarithms

The basic insight that makes working with logarithms interesting is that when you multiple or divide numbers expressed as powers of a common base, you can just add or subtract the exponents. Let's examine this in more detail. Let's say that we have some numbers that are powers of 2, such as 4 and 8. 4 can be expressed as 2^2 and 8 can be expressed as 2^3. If we want to multiply these numbers, we can express this as

$$y = 4 * 8 = 2^2 * 2^3$$

We know from the law of exponents that when we multiply two numbers which have the same base, we just add the exponents.

$$y = 4 * 8 = 2^{2+3} = 2^5 = 32$$

If we were to divide 4 by 8, we'd subtract the exponents.

$$y = 4/8 = 2^2/ 2^3 = 2^{2-3} = 2^{-1} = 1/2$$

In the general case, we can express this as

$$y = b^x * b^z = b^{x + z}$$

and

$$y = b^x/b^z = b^{x - z}$$

So how do we express all this in logarithms? Let's go back to our original example where we used powers of 2, and look at the number 8.

$$y = 8 = 2^3$$

If we were to take the log to the base 2 of both sides, this is what we'd get.

$$\log_2 y = \log_2 8 = \log_2 2^3$$

By taking the log to the base 2, we're asking, "What power do we have to raise 2 to to equal 8?" We know the answer is 3 but let's look at the algebraic manipulations to get to that answer. One "rule" of logarithms is that the log of a number to a power is equal to the power times the log. So this gives us

$$\log_2 8 = 3 \log_2 2$$

Our task now is to evaluate $\log_2 2$. The meaning of this is, "What power must 2 be raised to to equal 2?" The answer is 1 because $2^1 = 2$. So our result is

$$\log_2 8 = 3$$

There's something very important that needs to be pointed out here. To a very large degree, *it doesn't matter what the base b is*. Obviously, *b* cannot be zero or one. A practical system requires that *b* be a positive real number greater than 1. One of the most common values of *b* is 10. Now, let us explore common (base ten) logarithms a bit more.

The definition of a common logarithm is

$$y \equiv 10^{\log y}$$

That is, the common logarithm of a number is the power that 10 must be raised to to equal that number. The logarithm of 10 is 1 because $10^1 = 10$. The logarithm of 2 is about 0.301 because $10^{0.301} = 2$ (approximately).

Computing the Value of Common Logarithms

John Napier developed the concept of logarithms, and published a description of them in 1614. However, Napier did not use 10 as the base of his logarithms. Henry Briggs, a friend of Napier, visited with Napier and they agreed that the base 10 would be a better choice. Briggs began calculating the base 10 logarithms, publishing common logarithms for the numbers 1 to 1,000 in 1617. The question we address here is, "How did Briggs calculate these logarithms?"

The process of taking square roots was well known by Briggs and others. A very quick way to take a square root is the following iterative technique. Let *N* be the number you want to take the square root of. Let a_0 be your first guess of the square root (you can guess very roughly – the process will converge rapidly). Your next guess should be computed based on the following equation.

$$a_1 = 0.5 * (a_0 + N/ a_0)$$

For N = 10 and a first guess of $a_0 = 3$, we find the following results.

x	a_x	a_x^2
0	3	9
1	3.166666667	10.02778
2	3.162280702	10.00002
3	3.16227766	10

Table 1: Convergence towards a square root. The process converges very rapidly.

So Briggs could compute $10^{0.5}$ and any subsequent square roots ($10^{0.25}$, $10^{0.125}$, etc.). Expressed in fractional form, he could compute 10 to the 1/2, 1/4, 1/8, 1/16, 1/32, etc.

Now, we can do these calculations much easier than Briggs by using a calculator or a spreadsheet, but let's examine the values that Briggs would have calculated.

s in fractional form	*s* in decimal form	10^s
1/1	1	10
1/2	0.5	3.16227766
1/4	0.25	1.77827941
1/8	0.125	1.333521432
1/16	0.0625	1.154781985
1/32	0.03125	1.074607828
1/64	0.015625	1.036632928
1/128	0.0078125	1.018151722
1/256	0.00390625	1.009035045
1/512	0.001953125	1.004507364
1/1,024	0.000976563	1.002251148
1/2,048	0.000488281	1.001124941
1/4,096	0.000244141	1.000562313
1/8,192	0.000122070	1.000281117
1/16,384	6.10352E-05	1.0001405485
1/32,768	3.05176E-05	1.0000702718
1/65,536	1.52588E-05	1.0000351353
1/131,072	7.62939E-06	1.0000175675
1/262,144	3.8147E-06	1.0000087837
1/524,288	1.90735E-06	1.0000043918
1/1,048,576	9.53674E-07	1.0000021959
1/2,097,152	4.76837E-07	1.0000010980
1/4,194,304	2.38419E-07	1.0000005490
1/8,388,608	1.19209E-07	1.0000002745
1/16,777,216	5.96046E-08	1.0000001372
1/33,554,432	2.98023E-08	1.0000000686
1/67,108,864	1.49012E-08	1.0000000343
1/134,217,728	7.45058E-09	1.0000000172

Table 2: Twenty-seven successive square roots of 10.

And once you have successive square roots, you can calculate many other roots. For example, if you need to know the value of $10^{0.75}$, you can use $10^{0.5} * 10^{0.25}$ or $10^{0.5+0.25}$. Other roots can be found by addition and subtraction of existing powers (multiplication and division of values of 10^s). Let's take a more complex example – find the common logarithm of 2.

This is equivalent to the following equation:

$$2 = 10^x$$

We want to solve for *x*. Now if we look in the table, under the 10^s column for the value just less than 2 we see that it's equal to the exponent 0.25, giving a value of 10^s of 1.77827941. We know that *x* is a bit more than 0.25 so let's subtract 0.25 from *x* and examine the remainder. Now, subtracting an exponent is equivalent to division of the 10^s values.

$$2/1.77827941 = 10^x/10^{0.25}$$

Note that $10^x/10^{0.25} = 10^{(x - 0.25)}$.

$2/1.77827941 = 1.12468265$

Now, we find the closest value for 10^s that does not exceed 1.12468265, which is 10^s of 1.074607828 ($s = 1/32$).

Rather than go through each of the calculations, I'm going to give you the exponent values.

$x = 1/4 + 1/32 + 1/64 + 1/256 + 1/4096$ (and we could keep going)

Converting to a common denominator gives.

$x = 1024/4096 + 128/4096 + 64/4096 + 16/4096 + 1/4096$

$x = 1233/4096$

$x = 0.301025391$

$2 = 10^{0.301025391}$ (well, actually 1.999996 – if we kept going we'd get closer to a more accurate value of 0.301029996)

So once Briggs had enough successive square roots of 10 (and the record indicates that he took 54 successive square roots), he was able to calculate essentially any number in terms of a power of 10.

But taking 54 successive square roots is a lot of work – it'd be nice if there were a quicker way, or at least a fairly accurate approximation. And Briggs (and others) found one. Let's see what they found.

Suppose we look at the fractional part of the value of 10^s, that is, subtract 1 from each value of 10^s and see what remains. Suppose we further divide the fractional part by s, the exponent. Let's see what happens.

Logarithms of larger and smaller numbers

Once we have the log of 2, we can use that result to obtain the logarithms of larger numbers. For example, suppose we wanted to take the log of 20. 20 can be expressed as

$20 = 10 * 2$

or as powers of 10

$20 = 10^1 * 10^{0.30102}$

$20 = 10^{1+0.30102}$

$20 = 10^{1.30102}$

So the log of 20 is 1.30102…

This is the primary advantage of logarithms to the base 10. They allow you to easily find the logarithms of numbers larger than 10 or smaller than 1 (but >0).

s as a fraction	*s* as a decimal	10^s	10^s - 1	$(10^s – 1)/s$
1/1	1	10	9	9
1/2	0.5	3.16227766	2.16227766	4.32455532
1/4	0.25	1.77827941	0.77827941	3.11311764
1/8	0.125	1.333521432	0.333521432	2.668171457
1/16	0.0625	1.154781985	0.154781985	2.476511755
1/32	0.03125	1.074607828	0.074607828	2.387450506
1/64	0.015625	1.036632928	0.036632928	2.34450742
1/128	0.0078125	1.018151722	0.018151722	2.32342038
1/256	0.00390625	1.009035045	0.009035045	2.3129714794
1/512	0.001953125	1.004507364	0.004507364	2.3077704983
1/1,024	0.000976563	1.002251148	0.002251148	2.3051758519
1/2,048	0.000488281	1.001124941	0.001124941	2.3038799870
1/4,096	0.000244141	1.000562313	0.000562313	2.3032324186
1/8,192	0.000122070	1.000281117	0.000281117	2.3029087255
1/16,384	6.10352E-05	1.0001405485	0.0001405485	2.3027469017
1/32,768	3.05176E-05	1.0000702718	0.0000702718	2.3026659954
1/65,536	1.52588E-05	1.0000351353	0.0000351353	2.3026255437
1/131,072	7.62939E-06	1.0000175675	0.0000175675	2.3026053183
1/262,144	3.8147E-06	1.0000087837	0.0000087837	2.3025952056
1/524,288	1.90735E-06	1.0000043918	0.0000043918	2.3025901493
1/1,048,576	9.53674E-07	1.0000021959	0.0000021959	2.3025876211
1/2,097,152	4.76837E-07	1.0000010980	0.0000010980	2.3025863571
1/4,194,304	2.38419E-07	1.0000005490	0.0000005490	2.3025857247
1/8,388,608	1.19209E-07	1.0000002745	0.0000002745	2.3025854081
1/16,777,216	5.96046E-08	1.0000001372	0.0000001372	2.3025852516
1/33,554,432	2.98023E-08	1.0000000686	0.0000000686	2.3025851697
1/67,108,864	1.49012E-08	1.0000000343	0.0000000343	2.3025851399
1/134,217,728	7.45058E-09	1.0000000172	0.0000000172	2.3025851250

Table 3: Twenty-seven successive square roots of 10 and the fractional part divided by s.

Based on these calculations, we can see that for small values of s, the value of 10^s will equal approximately 1 + 2.3026s. So once Briggs got past, say, the first 27 successive square roots, he could calculate the logarithm, to a good level of accuracy, for each of the next 27 successive square roots by just computing 1+2.3026s.

Now, we have been using 10 as the basis for our logarithms but we now ask ourselves if there's a better, more natural, base for our logarithms. Specifically, we'd like to find a base, b, that allows us to approximate b^s as 1 + s, for small s. For those doing calculations back in the 17th century, this might make their work simpler.

But they would still want to obtain logarithms to the base 10. If we calculated the logarithms to a different base, can we convert them to another base, for example base 10? Suppose we have logarithms to the base b and want to create logarithms to the base x. How can we do that?

Earlier, I gave the definition of a common logarithm, which was

$y = 10^{\log y}$

Expressing this in general form,

$c = b^{\log c}$

Now, we want to compute the log of *c* to the base *x*. Or,

$c = x^{\log c}$

Let's take the log to the base *b* of both sides.

$\log_b c = \log_b (x^{\log c})$

Since the log of a number to a power is equal to the power times the log,

$\log_b c = \log_x c * \log_b x$

Since what we want is the log of *c* to the base *x*, we solve for that.

$\log_x c = (\log_b c)/\log_b x$

The answer is that we divide all our existing base *b* logarithms by $\log_b x$. A different, longer argument showing the same result is given in Appendix A.

Let me give a specific example of this logarithmic base conversion. Suppose we wanted to convert from logarithms to the base 10 to logarithms to the base 2. Let's take a simple base 10 log and convert it base 2. From our calculations above, we know that the logarithm of 3.1627766 is about 0.5 (this was the first square root of 10 that we calculated). We also know that the log of 2 to the base 10 is about 0.301029996. According to our calculations, if we divide the log of 3.1627766, which is 0.5, by 0.301029996, we will have the log of 3.1627766 to the base 2.

$\mathrm{Log}_2\ 3.1627766 = \mathrm{Log}_{10}\ 3.1627766 / \mathrm{Log}_{10}\ 2$

$\mathrm{Log}_2\ 3.1627766 = 0.5/0.301029996$

$\mathrm{Log}_2\ 3.1627766 = 1.660964046$

Which is quite close the actual value (the difference is due to the limited number of decimals in the value 3.1627766. The common log of that specific number is slightly greater than 0.5)

Thus, once you have one set of logarithms, you can convert the logarithms to any other base, simply by dividing them by a constant. So if we calculate our logarithms for some base other than 10, we can easily convert them to base 10, if we later want to.

Natural Logarithms

Earlier, I asked if there was a base that would allow us to estimate the logarithms of small exponents as 1+ s, rather than 1 + 2.3026s. But to make the terminology the same as used in the discussion above on changing the base of logarithms, and in Appendix A, let's change our terminology. For the (1 + 2.3026s), we're going to use (1 + 2.3026*a*), that is, *a* is the original logarithm ($a = \log_{10} c$). The (1 + s) term we will now call (1 + y) meaning that *y* is the logarithm to the new base ($y = \log_x c$). We can convert from *a* to *y* by using the following equation.

$$a = y/2.3026$$

You can see this by substituting for *a* in the 1 + 2.3026a equation – we get 1 + y. Rearranging the equation gives:

$$y = a * 2.3026$$

Now we remember, to convert logarithms from one base, *b* (and in this example, b = 10), to another base, x, we use this equation.

$$\log_x c = \log_b c/\log_b x$$

Substituting b = 10,

$$\log_x c = \log_{10} c/\log_{10} x$$

Earlier, we defined $y = \log_x c$ and $a = \log_{10} c$. Therefore,

$$y = a/\log_{10} x$$

And a few equations ago, we defined *y* to be

$$y = a * 2.3026$$

Equating the two equations

$$a * 2.3026 = a/\log_{10} x$$

Dividing both sides by *a*,

$$2.3026 = 1/ \log_{10} x$$

$\log_{10} x = 1/2.3026 = 0.43429167$

So what we want to know is what number is represented by

$x = 10^{0.43429167}$

We can determine this number by repeated processing against Table 2 or Table 3, but I'm going to take the easy way and compute it on a calculator or spreadsheet. The result is:

$x = 2.71826423$

This number[1] is known as the natural number, and is represented by the letter *e*. In logarithms, this base is known as the natural base[2], or base *e*. The definition of e is given by the following equation, as *x* goes to infinity.

$$e = \lim_{x\to\infty}(1+\tfrac{1}{x})^x$$

Now, *e* is just a number, but like π (which is also just a number), it has some very important properties, some of which are explored in the paper "Differential Equations for High School Students."

Computing e with an infinite series

e can also be computed with the infinite series

e = 1 + 1/1! + 1/2! + 1/3! + 1/4! + …
(4! = 4 * 3 * 2 * 1)

This is an excellent way to calculate the value of e quickly and accurately. The "compound interest" equation given in the text will usually give an incorrect result beyond 5 to 6 digits when calculated with a spreadsheet or calculator because of rounding errors and the large powers. When using an Excel spreadsheet, for example, thirteen terms of the above series gives the result 2.7182818284 which is accurate to 10 decimal places.

Sixteen terms produces an answer accurate to 13 decimal places. 2.7182818284590

[1] Actually, this value is not the real value of *e*. This value is just an approximation.
[2] Natural logarithms are written ln *x* instead of log *x*, the n in ln indicating "natural".

Appendix A – Converting the base of logarithms

Logarithms can be converted from one base to another quite easily. For example, suppose we have a base *b* and want to calculate the log of some value *c*. We write this as follows.

$c = b^a$ or $c = b^{\log_b c}$ $(a = \log_b c)$

Which means that *a* is the logarithm of *c* to the base *b*. Suppose that we now want to calculate the logarithm of *c* to some other base, which we'll call *x*.

$c = x^y$ or $c = x^{\log_x c}$ $(y = \log_x c)$

Here, we say that *y* is the logarithm of *c* to the base *x*. How is y related to *a*? Let's investigate. Since we're assuming that both *b* and *x* are real numbers, each of a constant value, we can express *x* in terms of *b*.

$x = b^k$ or $x = b^{\log_b x}$ $(k = \log_b x)$

Note that *k* is the logarithm of *x* to the base *b*. Let's substitute for *x* in the earlier equation, $c = x^y$.

$c = (b^k)^y$ or $c = (b^{\log_b x})^{\log_x c}$

$c = b^{ky}$ or $c = b^{(\log_b x * \log_x c)}$

But remember the equation we started off with?

$c = b^a$ or $c = b^{\log_b c}$

Now, we substitute for *c*.

$b^a = b^{ky}$ or $b^{\log_b c} = b^{(\log_b x * \log_x c)}$

Which implies

$a = ky$ or $\log_b c = \log_b x * \log_x c$

Solving for *y*

$y= a/k$ or $\log_x c = \log_b c/\log_b x$

So to convert from one base to another, we simply divide our existing logarithms by a constant, which is the logarithm of the number we use for the new base, to the old base. Specifically, we divide all our existing base *b* logarithms by $\log_b x$.

Appendix B – Logarithms of Negative Numbers

When I discussed logarithms earlier, I was careful to avoid situations where we might have to take the log of a negative number. But is it possible to take the log of a negative number? Let's see.

When we take the log of a positive real number, what we're asked to do is to find a number that 10 can be raised to to give that number. So if we were asked to find the log of 5, we'd ask, "What power can 10 be raised to to give the value 5?" The answer is about 0.6989

Now, suppose we were asked to take the log of -5. What we're asked to do is to find a number that 10 can be raised to to give the value of -5. But there's no real number that will do that.

Perhaps there's a complex number that will solve this problem.

$$\text{Log}_{10}(-5) = ?$$

Since we're going to work with complex numbers, let's convert to the natural log.

$$\text{Log}_{10}(-5) = \frac{\ln(-5)}{\ln 10}$$

Since -5 = 5*(-1) we can rewrite this as

$$\text{Log}_{10}(-5) = \frac{\ln(5)+\ln(-1)}{\ln 10}$$

We can compute ln (5) so our challenge is to evaluate ln (-1). Remember from the Differential Equations paper

$$e^{i\pi} = -1$$

Therefore

$$\ln(e^{i\pi}) = \ln(-1)$$

But we know that the natural log of e is equal to the exponent so

$$i\pi = \ln(-1)$$

And

$$\text{Log}_{10}(-5) = \frac{\ln(5)+i\pi)}{\ln 10}$$

Giving

$$\text{Log}_{10}(-5) = 0.69897 + 1.36437i$$

In the general case, for logs other than base 10,

$$\mathrm{Log}_b\,(-x) = \frac{\ln(x)+i\pi)}{\ln b}$$

Bibliography

Dunham, William, *Euler, The Master of Us All*. The Mathematical Association of America, 1999.

Feynman, Richard P., Robert B. Leighton, and Matthew Sands, *The Feynman Lectures on Physics, Volume 1*. Addison-Wesley Publishing, Reading, MA, 1963. (See especially chapter 21, "The Harmonic Oscillator" and chapter 22, "Algebra". Feynman comments in a later volume that he was disappointed with the test results of the students who took his introductory physics class at Caltech. Read this book thinking about how you would have fared if you had walked into Feynman's lectures right out of high school.)

Gullberg, Jan, *Mathematics: From the Birth of Numbers*. W. W. Norton, 1997. (Excellent book, recommended to anyone with an interest in mathematics.)

Maor, Eli, *e: The Story of a Number*. Princeton University Press, 1994. (Good book on *e*, understandable to anyone with high school mathematics.)

Nahin, Paul J., *An Imaginary Tale. The Story of* $\sqrt{-1}$. Princeton University Press, 1998. (This book is not a good introduction to imaginary numbers – it just requires too much knowledge of mathematics to be useful to a high school student.)